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Quantum Dynamics of a Nonlinear Kicked Oscillator

S. Mukhopadhyay [1], B. Demircioglu [2] and A. Chatterjee [3*]

[1] *Dr. V. R. K. Women's College of Engineering and Technology, Aziznagar, Moinabad, Hyderabad, India*
[2] *Saraykoy Nukleer Arastirma ve Egitim Merkezi Saray Mahallesi, Atom Caddesi, No:27, 06983 Saray-Kazan-Ankara, Turkey*
[3] *School of Physics, University of Hyderabad, Hyderabad, India*

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Abstract: The quantum dynamics of a nonlinear kicked oscillator is studied by a recently proposed complex non-hermitian Hamiltonian technique. It is shown that the probability density and the energy function display either a growing or a decaying exponential time dependence characteristic of absorption or dissipation. It is furthermore shown that though a decrease in the kicking period increases the diffusive motion leading to the ballistic spreading, increasing its value does not apparently favour any localization. The anharmonicity also enhances the dissipative dynamics but with time gives rise to energy crossings typical of a quantum chaos. The variation in the spatial periodicity of the delta-function kicking however exhibits a more complex behaviour showing diffusive character to super-diffusion leading to ballistic motion on the one side and the quantum localization on the other.

Keywords: *nonlinear kicked oscillator; quantum diffusion; dissipation; localization*

Mathematics Subject Classification (2000): 35Q72, 81Q50, 37L50.

1 Introduction

Recent years have witnessed a flurry of investigations in the area of quantum chaos and dynamical quantum localization and in this context the quantum dynamics of area preserving maps has attracted a particular attention [1, 2]. The kicked harmonic oscillator is an example that belongs to this class and has been studied quite extensively in the last two decades [3, 4, 5, 6]. The kicked harmonic oscillator however has generated some renewed interest in recent times for it simulates some interesting low-dimensional systems like quantum wires, semiconductor superlattices [7] or trapped ions [8] periodically

* Corresponding author: mailto:acsp@uohyd.ernet.in

kicked by intense monochromatic short-time pulses like laser light. The effect of Gross–Pitaevskii nonlinearity in the kicked oscillator has been recently studied by Artuso and Rebuzzini [9] and they have shown that the qualitative features depend strongly on the parameters of the system.

The dynamics of a kicked oscillator system can be determined by studying the generalized master equation for the probability distribution or by calculating the propagator and thereby constructing the time-dependent wave function using the propagator as the kernel for the time development. Of course the probability density obtained from these methods has to be consistent with the time-dependent Schrödinger equation. Several techniques have been proposed in recent years to deal with the nonlinear differential equations with varying degree of success and rigour. Kovalev et al. have developed the method of oriental manifold to study the geometric properties of nonlinear differential systems with control [10]. Stability of systems with linear and nonlinear perturbations has been studied by Jeffrey J. DaCunha [11]. Practical stability and controllability for a class of nonlinear discrete systems with time delay have been investigated by Su et al. [12].

Recently Liboff and Porter [13] have proposed a novel technique to study the energy absorption and dissipation in quantum systems by introducing a complex non-Hermitian term in the Hamiltonian. The purpose of the present paper is to apply this technique to study the dynamics of a kicked oscillator problem. Since the trapping potential of the real ion traps or the ion-ion potential in a quantum wire may not be strictly harmonic, we introduce the anharmonicity in the problem and study its effect on the oscillator dynamics. We also include a spatial periodicity in the kicking term. It must however be mentioned that the nonlinearity considered here is different from that studied in [9]. We consider the oscillator to be anharmonic, while in [9] the potential is nonlinear in the wavefunction itself and therefore the problem of [9] requires a self-consistent solution. Our problem will be important for a quantum wire like a carbon nanotube with an anharmonic confining potential and periodically kicked by a laser wave. We find, as expected from [13], that the probability density and the energy of a kicked nonlinear oscillator exhibit as a function of time a growing or a decaying behaviour depending on the sign of the coefficient that gives the kicking strength. We observe that as the kicking period is decreased, the motion of the nonlinear oscillator becomes more and more diffusive finally reaching the ballistic regime. However, interestingly enough, increasing the kicking period does not apparently yield any localization. The anharmonicity is also found to favour the dissipative dynamics and gives rise to crossing of the energy curves characteristic of quantum chaos. We furthermore show that the spatial periodicity of the kicking potential seems to be a very sensitive parameter the variation of which can lead to a variety of features ranging from a ballistic motion through classical diffusion to dynamical localization. In what follows we shall first briefly discuss the method of Liboff and Porter [13] and then apply it to the problem of a nonlinear kicked oscillator.

2 General Formalism of Liboff and Porter

Consider a complex Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(x) + i\hbar\alpha(x,t) = H_0 + i\hbar\alpha(x,t), \tag{1}$$

where $\alpha(x,t)$ can be written as a product of the space-part and the time-part and the unperturbed Hamiltonian satisfies the time-independent Schrödinger equation

$$H_0 u_n(x) = E_n^{(0)} u_n(x). \tag{2}$$

We can therefore write

$$[H_0 + i\hbar\alpha(x,t)]u_n(x) = E_n u_n(x) = [E_n^{(0)} + i\hbar\alpha(x,t)]u_n(x). \tag{3}$$

The time-dependent Schrödinger equation for the problem is given by

$$i\hbar\partial\Psi/\partial t = H\Psi = [H_0 + i\hbar\alpha(x,t)]\Psi, \tag{4}$$

where $\Psi(t)$ can be formally written as

$$\Psi(t) = exp\ [-\frac{i}{\hbar}\int_0^t d\lambda H(\lambda)]\Psi(0), \tag{5}$$

where $\Psi(0)$ is the initial state function given by

$$\Psi(0) = \sum_n a_n u_n. \tag{6}$$

The expectation value of the Hamiltonian at $t = 0$ yields

$$< \Psi(0)|H|\Psi(0) >= \sum_n |a_n|^2 E_n^{(0)} = E_0, \tag{7}$$

where the expansion coefficients may be fixed from the knowledge of the initial configuration of the state of the system. Substituting (6) in (5) and using the eigenvalue equation (3), we get

$$\Psi(t) = e^{g(t)} \sum_n a_n u_n e^{-\frac{i}{\hbar}E_n^{(0)}t}, \tag{8}$$

where

$$g(t) = \int_0^t dt\alpha(t), \tag{9}$$

so that the real part of the energy expectation value at time t $(E(t))$ is given by

$$E(t) = Re < \Psi(t)|H|\Psi(t) >= E_0 e^{\,2\int_0^t d\lambda\alpha(\lambda)}. \tag{10}$$

3 Kicked Harmonic Oscillator

We shall now employ this formalism to a kicked nonlinear oscillator for which we write

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \lambda x^4 \tag{11}$$

and choose $\alpha(x,t)$ as

$$\alpha(x,t) = -\varepsilon < \cos(kx) > \sum_{s=1}^{N} \delta(t - Ts), \tag{12}$$

where ϵ gives the measure of the δ-function kicking, k measures the spatial periodicity of the kicking potential, T is the kicking period, N is the number of kicks and $< \cos(kx) > \equiv \kappa$ is the expectation value of $\cos(kx)$ taken with respect to the eigenstate of the effective harmonic oscillator. We assume that the initial state is prepared in the n-th excited state of the linear oscillator and incorporate the quartic term using a mean-field approximation so that the unperturbed potential can be written as an effective harmonic oscillator with a new frequency

$$\tilde{\omega} = [\omega^2 + \frac{2\lambda}{m} < x^2 >]^{1/2}, \tag{13}$$

where

$$< x^2 >= \int_{-\infty}^{\infty} u_n^{(0)} x^2 u_n^{(0)} dx. \tag{14}$$

According to our initial configuration,

$$< x^2 >= (n + \frac{1}{2}), \tag{15}$$

and hence

$$\tilde{\omega} = [1 + 2\lambda(n + \frac{1}{2})]^{1/2}, \tag{16}$$

where we have assumed $m = \hbar = \omega = 1$. A simple calculation shows that κ can be obtained as

$$\kappa = e^{-\beta^2} L_n(\beta^2), \tag{17}$$

where $\beta = (1/2\tilde{\omega})^{1/2} k$ and $L_n(x)$ is the Laguerre polynomial of order n and is given by

$$L_n(x) = \sum_{m=0}^{n} (-1)^m \frac{n! \beta^{2m}}{(m!)^2 (n-m)!}. \tag{18}$$

The time-dependent energy $E(t)$ is then finally obtained as

$$E(t) = E_0 P(t), \tag{19}$$

where $P(t)$ is the temporal probability that the system would be found in the state $\Psi(t)$ at time t and is given by

$$P(t) =< \Psi(t) | \Psi(t) >= e^{-2\varepsilon\kappa\phi(t)}, \tag{20}$$

where

$$\phi(t) = \sum_{s=1}^{N} \int_0^t \delta(\lambda - sT) d\lambda. \tag{21}$$

One can immediately see that $\phi(t)$ is equal to the number of s values for which s is less than t/T. We would like to point out here that in [10] the value of $\phi(t)$ has been determined erroneously. In fact the definition of $\phi(t)$ in [10] violates causality. We obtain

$$P_l[lT \leq t < (l+1)T] = e^{-2\varepsilon\kappa l}, \tag{22}$$

where $l = 0, 1, 2, \ldots$, and consequently the time-dependent energy of the n-th excited state of the nonlinear oscillator reads

$$E_n[lT \leq t < (l+1)T] = (n + \frac{1}{2}) e^{-2\varepsilon\kappa l}. \tag{23}$$

4 Numerical Results and Discussion

In Figure 1, we show the behaviour of the energy of the first excited state as a function of time. In (12), we have chosen the sign of the kicking term as negative in order that the dynamics becomes dissipative. We show the energy dissipation with time for three values of the kicking strength ε. One can observe that the delta-function kicking results in instantaneous dissipations in the energy at the kicking times and remains constant in between two successive kicks leading to a stair-case like structure. As the kicking strength increases, the dissipation of course becomes more and more rapid.

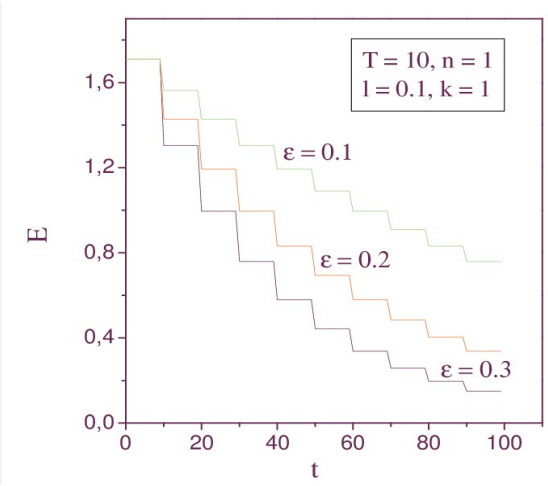

Figure 1: The first excited state energy E as a function of time for three values of the kicking strength ε.

In Figure 2 we compare the dissipation for the first two excited states ($n = 1$ and $n = 2$) of the nonlinear oscillator. The dissipation behaviour of the second excited state is almost similar to the first one, except that the decay rate is faster for the second excited state than that for the first excited state. In Figure 3 we study the energy-time behaviour for the first excited state for a few values of the nonlinearity parameter λ. We find that the energy, as expected for the present system, is initially larger for a larger value of λ but the decay is interestingly faster for larger λ-values. This leads to an interesting crossing of the energies at a long enough time which may be attributed to a dynamics akin to quantum chaos.

The variation of the dissipative dynamics as we change the kicking period T is studied in Figure 4. It is clearly evident that the dissipation becomes more and more rapid as the kicking period decreases so much so that for very low values of T, the dynamics is essentially ballistic, while for very large values of T the energy spreading is more or less diffusive. However we do not observe any localization here.

From (17) and (22) one can note that neither the temporal probability $P(t)$ nor the energy E(t) is a monotonically increasing or decreasing function of k. In fact the dependence of E on k is quite interesting which we show in Figure 5 where we have plotted the energy for the first two excited states as a function of k for three values of time t. One can see that the first excited state energy has two maxima, lying symmetrically

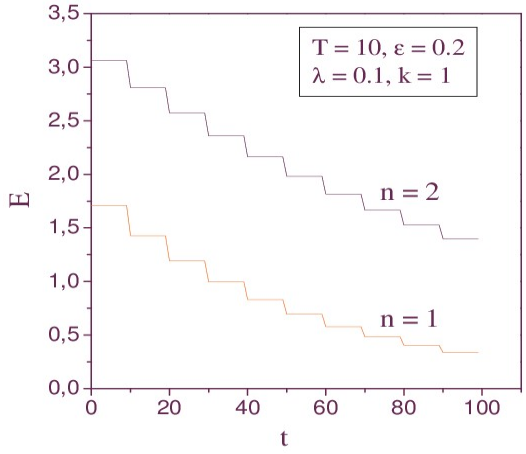

Figure 2: The decay of the energies of first two excited states $n = 1$ and $n = 2$ with time.

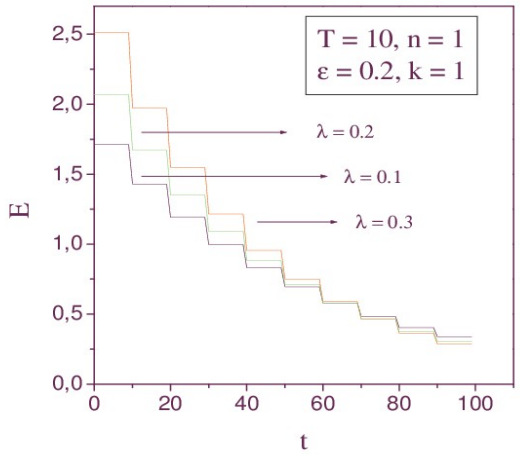

Figure 3: The dissipation of the first excited state with time for three values of the nonlinearity parameter λ.

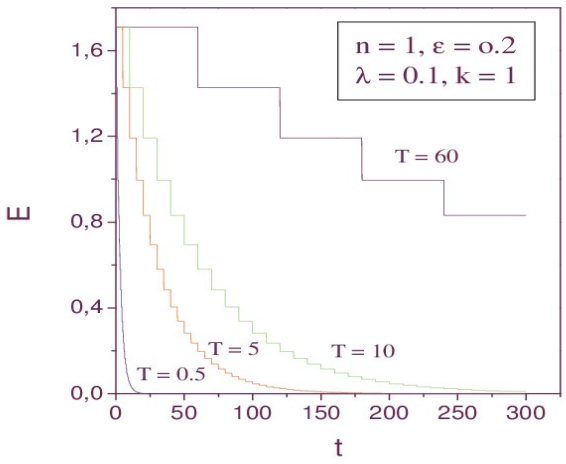

Figure 4: The dissipation of the first excited state with time for four values of the kicking period T.

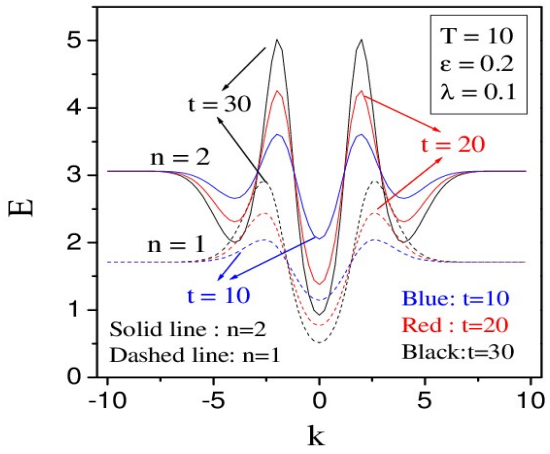

Figure 5: The variation of the time-dependent energy of the first two excited states as a function of the spatial periodicity k of the kicking potential for three values of time t.

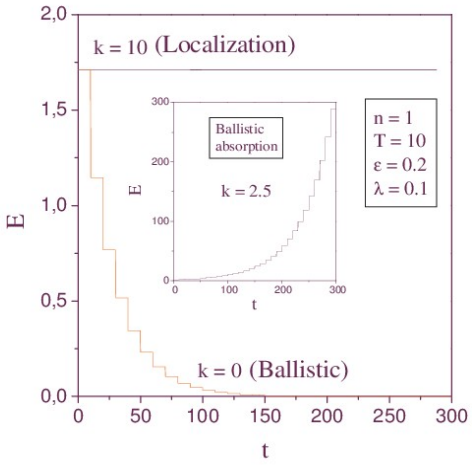

Figure 6: The variation of the energy of the first excited state as a function of time t for $k = 0$ and $k = 10$. The inset shows the behaviour for $k = 2.5$.

about $k = 0$ where it has a central minimum and the energy saturates to a constant after k reaches some critical value. Interestingly, however, the locations of the maxima and minima on the k-line do not change with time. For $n = 2$, however, two additional (secondary) minima develop symmetrically on either side of $k = 0$ and the maxima get shifted outward on both sides of $k = 0$, while the central minima still lie at $k = 0$.

The maxima-minima structure seems to have an interesting bearing on the dynamical behaviour of the system and throws up a variety of possibilities for different ranges of the k-values. For $n = 1$, with the parameter values we have chosen, the maxima occurs at around $k = \pm 2.5$. The central minimum for all the cases however occur at $k = 0$ as has already pointed out. This value of $k = 0$ gives dissipative dynamics in the ballistic regime as shown in Figure 6. It is quite clear from Figure 5 that for large values of k the system would exhibit localization. We have confirmed this behaviour by plotting E as a function of t for $k = 10$ in Figure 6. Interestingly however, $k = 2.5$ for the ground state corresponds to ballistic absorption. In the inset of Figure 6 we display this behaviour.

In Figure 7 we show in more detail the dissipative behaviour of the system for different values of k. For the value of k close to 1.48, one can observe that the dynamics is more or less diffusive and for lower values of k it becomes more and more super-diffusive and finally reaches almost the ballistic limit, while for about $k = 1.51$, the system shows a dynamical localization.

5 Conclusion

In conclusion, we have studied the dynamics of a nonlinear oscillator kicked by a time-periodic δ-function potential that has a spatial periodicity of the cosine-form using a complex nonhermitian Hamiltonian technique recently proposed by Liboff and Porter [13]. We have observed that the system can exhibit exponential growth or decay depending on the sign of the kicking term and the value of the spatial periodicity parameter. In particular, we have studied the case of dissipation and have shown that it increases

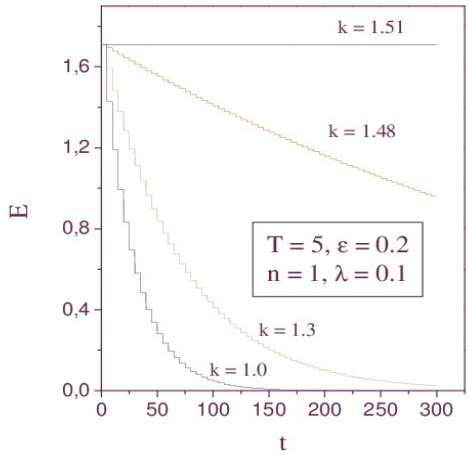

Figure 7: The variation of the energy of the first excited state as a function of time t for $k = 1.0$, $k = 1.3$, $k = 1.48$ and $k = 1.51$.

with increasing kicking strength. The dissipation however takes place instantaneously with kicking and energy remains constant between two successive kicking leading to a stair-case structure. As a function of the nonlinearity parameter, there occurs a crossing of the energy curves that seems to characterize the onset of a quantum chaos. However, the system never shows any indication of localization for any value of the kicking period that we have considered. Rather it exhibits a more and more diffusive behaviour as the kicking period decreases reaching finally the ballistic limit. Most interestingly, the dynamics of the system seems to depend quite sensitively on the spatial periodicity parameter, the variation of which gives rise to a variety of rich phenomena ranging from diffusive to super-diffusive behaviour to ballistic spreading on the one side and to dynamical quantum localization on the other.

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