Nonlinear Dynamics and Systems Theory, **10** (4) (2010) 325–337

Liapunov Functionals, Convex Kernels, and Strategy

T.A. Burton

Northwest Research Institute
732 Caroline St., Port Angeles, WA 98362

Received: October 28, 2009; Revised: October 13, 2010

Abstract: We study an integral equation of the form $x(t) = a(t) - \int_0^t C(t,s)g(x(s))ds$ where C is convex and g has the sign of x. In earlier work we treated the case of $\sup \int_s^t C^2(u,s)du =: \Gamma < \infty$. Here, we study the case of $\Gamma = \infty$ by looking at a new equation formed from $x' + kx$ with k a positive constant. This enables us to define a Liapunov functional which will give a bound on $\int_0^t g^2(x(s))ds$ and a parallel bound on one of the resolvents in the linear case. Equations of this type have been used since the early work of Volterra in a number of real-world problems.

Keywords: *integral equations; Liapunov functionals; resolvents.*

Mathematics Subject Classification (2000): 47G05, 34D20.

1 Introduction

We are concerned here with an integral equation

$$x(t) = a(t) - \int_0^t C(t,s)g(x(s))ds, \tag{1}$$

where $a : [0,\infty) \to \Re$ is continuous, while C is continuous for $0 \le s \le t < \infty$, and $g : \Re \to \Re$ is continuous with $xg(x) > 0$ if $x \ne 0$. Continuity of a, C, g will ensure the existence of a solution. If the solution remains bounded, then it can be continued on $[0,\infty)$. See [5; pp. 178-180], for example.

It is always assumed that the kernel, $C(t,s)$, is convex in the sense that

$$C(t,s) \ge 0, \quad C_s(t,s) \ge 0, \quad C_{st}(t,s) \le 0, \quad C_t(t,s) \le 0. \tag{2}$$

Convolution problems of this type are seen in Levin [10] and Londen [11], for example.

* Corresponding author: mailto:taburton@olypen.com

In the classical theory of integral equations we generally need to ask that the kernel be very small in order to obtain global stability results. In 1928, Volterra [13] noticed that a great many real world problems were being modeled by integral and integro-differential equations with convex kernels which inherently suggested a fading memory. He conjectured that there is a Liapunov functional for such kernels which would yield much qualitative information about solutions and which would allow very large kernels. Today we see problems in biology, nuclear reactors, viscoelasticity, and neural networks being modeled using convex kernels.

In 1963, Levin followed Volterra's idea and constructed such a Liapunov functional for a convolution form of the integro-differential equation

$$x' = -\int_0^t C(t,s)g(x(s))ds$$

with C convex and in 1992 [2] we constructed one for integral equations in the form of (1). For the linear integral equation there is also a Liapunov functional for the resolvent equation and we discussed this in some detail in [6] when $\sup_{0\le s\le t<\infty}\int_s^t C^2(u,s)du =: \Gamma < \infty$. This paper seeks to extend some of that work to the case $\Gamma = \infty$. In the nonlinear integral equation there is a severe technical problem in dealing with the derivative of the Liapunov functional and the investigator must make some undesirable assumptions about the nonlinearity. This paper offers an alternative to those assumptions. Here is some detail concerning the two difficulties which we study.

Our Liapunov functional

$$V_1(t) = \int_0^t C_s(t,s)\bigg(\int_s^t g(x(u))du\bigg)^2 ds + C(t,0)\bigg(\int_0^t g(x(u))du\bigg)^2$$

for (1) has a derivative satisfying

$$V_1'(t) \le 2g(x)[a(t) - x(t)].$$

Owing to the absence of a chain rule, that differentiation is not simple so we want to give the details. It would be a distraction to give them here, so we offer them in the appendix.

In order to relate $g(x)$ to $a(t)$ we need to be able to separate that relation into

$$V_1'(t) \le |p(a(t))| - |q(x(t)|$$

for some functions p and q with q positive definite with respect to x or $g(x)$ and p positive definite with respect to $a(t)$ so that

$$0 \le V_1(t) \le V_1(0) + \int_0^t |p(a(s))|ds - \int_0^t |q(x(s))|ds.$$

That separation has proved to be very cumbersome and investigators ([5; pp. 190-191], [4], [14]) have resorted to ad hoc assumptions, as well as stringent conditions on g in order to use Young's inequality. A definite example will show the need for the theory which is to follow.

Example 1.1 Consider the scalar equation

$$x(t) = a(t) - \int_0^t [1+t-s]^{-1/4} g(x(s))ds$$

where g is an arbitrary continuous function satisfying $xg(x) > 0$ if $x \neq 0$. For $a \in L^2[0,\infty)$ if x is a solution on $[0,\infty)$ then we know of no result or technique in the literature that will yield $g(x) \in L^p[0,\infty)$. The Liapunov functional mentioned above will yield the indicated derivative and we find no way to perform the required separation. The difficulty will vanish using Theorem 3.1, (14). We will immediately find $g(x) \in L^2[0,\infty)$ without further restriction on g.

In the linear case, $g(x) = x$, we have

$$V_1'(t) \le a^2(t) - x^2(t)$$

so that

$$\int_0^t x^2(s)ds \le \int_0^t a^2(s)ds,$$

a very useful relation. Moreover, it extends to the resolvent equation ([5; pp. 130-131])

$$R(t,s) = C(t,s) - \int_s^t C(t,u)R(u,s)du$$

as

$$V_2(t) = \int_s^t C_v(t,v)\bigg(\int_v^t R(u,s)du\bigg)^2 dv + C(t,s)\bigg(\int_s^t R(u,s)du\bigg)^2$$

with a derivative satisfying

$$V_2'(t) \le -R^2(t,s) + C^2(t,s)$$

as may be seen following the details in the appendix. This yields

$$\int_s^t R^2(u,s)du \le \int_s^t C^2(u,s)du$$

which is so useful in the variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds.$$

But we have a difficulty here also. If $\sup_{0\le s\le t<\infty} \int_s^t C^2(u,s)du =: \Gamma < \infty$ then we have a very useful parallel property for R. On the other hand, if $\Gamma = \infty$ then the property is lost and we are left with the obvious fact that if $a \in L^2$ then $x \in L^2$ so by default $\int_0^t R(t,s)a(s)ds \in L^2$ and $x - a \in L^2$, but we can not extract from that any essential properties of R itself.

Example 1.2 We can continue Example 1.1 with $g(x) = x$ and study $x(t) = a(t) - \int_0^t [1+t-s]^{-1/4}x(s)ds$. The Liapunov functional of the appendix will yield $\int_0^t x^2(s)ds \le \int_0^t a^2(s)ds$ and $\int_0^t R(t,s)a(s)ds \in L^2[0,\infty)$ when $a \in L^2$ without any independent property of R. Our second goal is to obtain basic properties of a resolvent independent of $a(t)$. That resolvent will not be R but it will serve in a parallel way to R.

Thus, we encounter fundamental problems in both the nonlinear and linear cases. These two unsolved problems will drive this paper.

In an effort to avoid the difficulties just mentioned we consider the old technique of differentiating (1) to obtain

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds$$

which seems promising since for $C(t,t) \geq \alpha > 0$ we have a perturbation of the uniformly asymptotically stable equation

$$x' + C(t,t)x = 0.$$

However, that gain pales in comparison to our great loss in that $C_t(t,s)$ is no longer convex; hence, we would require some restrictions on the magnitude of $C_t(t,s)$ in order to use standard results on qualitative properties. To avoid all of those problems we develop a strategy which yields very good results.

Moreover, there is an added benefit, uncommon in the theory of convex kernels. If we can find a function $f : [0,\infty) \to [0,\infty)$ with $\int_0^t \frac{ds}{f(s)}$ continuous for $t \geq 0$,

$$|g(x)| \leq f\left(\int_0^x g(s)ds\right),\ x \in \Re,\ f \in \nearrow, \tag{3}$$

and

$$\int_0^\infty \frac{ds}{f(s)} = \infty, \tag{4}$$

then we prove that the solution has certain integral properties.

The work is based on four Liapunov functionals, a differential inequality, and a strategy for finding a strongly stable equation which has a solution of (1) as one of its solutions.

2 The Strategy

We will now employ a very simple device which seems to have been totally overlooked in the literature until recently. It was introduced in Burton [5], further developed in Burton-Haddock [7], and has significant applications in the existence of periodic solutions, a project to be presented later.

A classic strategy is to differentiate (1), turning it into a differential equation

$$x'(t) = a'(t) - C(t,t)g(x) - \int_0^t C_t(t,s)g(x(s))ds. \tag{5}$$

Among other techniques, we can then apply Liapunov's direct method, as discussed in Miller [12; p. 337]. While it sometimes is effective, it is usually a disaster since differentiation tends to have a very non-smoothing effect. But under some general conditions, if $C(t,s)$ is convex and if k is a sufficiently large positive constant, then it is true that

$$D(t,s) := kC(t,s) + C_t(t,s) \tag{6}$$

is convex. For example, it is readily verified that if r is a positive constant, then $k = r+3$ is a suitable constant for $C(t,s) = [1+t-s]^{-r}$ (this pertains to Example 1.1), while $k = r+1$ is suitable for $C(t,s) = e^{-r(t-s)}$.

If we form $x' + kx$ then we have

$$x'(t) = a'(t) + ka(t) - [kx + C(t,t)g(x)] - \int_0^t D(t,s)g(x(s))ds. \tag{7}$$

This is a one-parameter family of totally different equations having exactly one property in common: a solution of (1) satisfies every one of those equations. If all solutions of (7) satisfy a certain property, so does a solution of (1). Two things have happened. Since C is convex, $C(t,t) \geq 0$ and, hence, $x' + kx + C(t,t)g(x) = 0$ is uniformly asymptotically stable. If $a'(t) + ka(t) \in L^2[0,\infty)$ and $C(t,t) \geq \alpha > 0$, then Levin's original Liapunov functional will yield $g(x(t)) \in L^2[0,\infty)$. In addition, if (3) and (4) hold and if $a' + ka \in L^1[0,\infty)$, then we will obtain an L^2 result for x.

We have used (5) to introduce a differential equation, but we have overwhelmed it with the integral equation by taking k large. In the parallel work with Haddock [7] the technique was different in that a very careful selection of an exact value for k was made. An entirely different selection is made in the aforementioned work with periodic solutions. But all three projects stem from the same idea.

What is, perhaps, more interesting is the fact that when $g(x) = x$, then Becker's [1] resolvent equation for (7) is

$$Z_t(t,s) = -[k + C(t,t)]Z(t,s) - \int_s^t D(t,u)Z(u,s)du \tag{8}$$

and a slight modification of Levin's [9] Liapunov functional will yield

$$\sup_{0\leq s\leq t<\infty} \int_s^t Z^2(u,s)du < \infty. \tag{9}$$

This is a critical result in the variation of parameters formula

$$x(t) = Z(t,0)x(0) + \int_0^t Z(t,s)[a'(s) + ka(s)]ds \tag{10}$$

where x solves (1) if $x(0) = a(0)$. These ideas will be developed in the coming sections.

3 The Nonlinear Problem

In 1963, Levin [9] considered a convolution form of

$$x' = -\int_0^t D(t,s)g(x(s))ds \tag{11}$$

with $xg(x) > 0$ for $x \neq 0$ and D convex. He constructed the Liapunov functional

$$V_3(t) = \int_0^x g(s)ds + \frac{1}{2}\int_0^t D_s(t,s)\left(\int_s^t g(x(u))du\right)^2 ds + \frac{1}{2}D(t,0)\left(\int_0^t g(x(u))du\right)^2 \tag{12}$$

and found that $V_3'(t) \leq 0$ along a solution of (11). This means that

$$\int_0^{x(t)} g(s)ds \leq V_3(t) \leq V_3(0) = \int_0^{x(0)} g(s)ds$$

so that if $\int_0^{\pm\infty} g(s)ds = \infty$, then every solution of (11) is bounded.

We are going to use the same Liapunov functional on (7). In (16) below recall that $xg(x) > 0$ if $x \neq 0$ and that $C(t,t) \geq 0$.

Theorem 3.1 *Suppose that D is convex,*

$$D(t,s) \geq 0, \quad D_s(t,s) \geq 0, \quad D_{st}(t,s) \leq 0, \quad D_t(t,s) \leq 0, \tag{13}$$

that $xg(x) > 0$ if $x \neq 0$, and that V_3 is defined in (12). Then along a solution of (7) we have

$$V_3'(t) \leq g(x)[a'(t) + ka(t)] - g(x)[kx + C(t,t)g(x)]. \tag{14}$$

If, in addition, $|kx + C(t,t)g(x)| \geq \mu|g(x)|$, for some $\mu > 0$, then $a' + ka \in L^2[0,\infty)$ implies $g(x(t)) \in L^2[0,\infty)$. In particular, any solution x in Example 1.1 satisfies $g(x) \in L^2[0,\infty)$.

If (14) and (3) hold then along a solution of (7) we have

$$V_3'(t) \leq f(V_3(t))|a'(t) + ka(t)|. \tag{15}$$

If, in addition, (4) holds, $\int_0^{\pm\infty} g(s)ds = \infty$, and $a' + ka \in L^1[0,\infty)$, then every solution of (7) is bounded and

$$\int_0^\infty g(x(s))[kx(s) + C(s,s)g(x(s))]ds < \infty. \tag{16}$$

Proof Along a solution of (7) we have

$$V_3'(t) \leq g(x)[a'(t) + ka(t)] - g(x)[kx + C(t,t)g(x)] - g(x)\int_0^t D(t,s)g(x(s)ds$$
$$+ g(x)D(t,0)\int_0^t g(x(u))du + g(x)\int_0^t D_s(t,s)\int_s^t g(x(u))duds.$$

Integrating the last term by parts yields

$$g(x)[D(t,s)\int_s^t g(x(u))du\Big|_0^t + \int_0^t D(t,s)g(x(s))ds]$$
$$= g(x)[-D(t,0)\int_0^t g(x(u))du + \int_0^t D(t,s)g(x(s))ds]$$

so that

$$V_3'(t) \leq g(x)[a'(t) + ka(t)] - g(x)[kx + C(t,t)g(x)].$$

Now, if $|kx + C(t,t)g(x)| \geq \mu|g(x)|$ then

$$V_3'(t) \leq \alpha[a'(t) + ka(t)]^2 - \beta g^2(x)$$

for some positive α and β, from which $a' + ka \in L^2$ implies $g(x) \in L^2$.

Remark 3.1 The obvious and usual condition is that $C(t,t)$ be greater than a positive constant, entirely consistent with the convexity. Indeed, in the convolution case $C(t) \geq 0$ and $C'(t) \leq 0$ so if $C(0) = 0$ then $C(t) \equiv 0$. Even if this fails, in the next step we get x bounded. None of the ad hoc assumptions on g needed in Young's inequality found in earlier work (e.g., [4]) are needed.

Next, if (14) and (3) hold, then

$$V_3'(t) \leq f\bigg(\int_0^x g(s)ds\bigg)|a'(t)+ka(t)| \leq f(V_3)|a'(t)+ka(t)|$$

so

$$\int_{V_3(0)}^{V_3(t)} \frac{du}{f(u)} \leq \int_0^t |a'(s)+ka(s)|ds.$$

If (4) holds and $a'+ka \in L^1$, then $V_3(t)$ is bounded and, hence, $x(t)$ is bounded. This means that $g(x)[a'+ka] \in L^1$ so from (14) we see that (16) follows. The proof is complete.

These results raise questions for the linear case. For we then see that $a'+ka \in L^1$ yields $x \in L^2$, but $a'+ka \in L^2$ also yields $x \in L^2$. Linear theory shows that $x \in L^1[0,\infty)$ is intimately related to uniform asymptotic stability. The next result shows that for certain choices of g we approximate $x \in L^1[0,\infty)$.

Theorem 3.2 *Suppose that D is convex and that $g(x) = x^{1/n}$ where n is an odd positive integer. If $a'+ka \in L^1[0,\infty)$, then $\int_0^\infty |x(s)|^{\frac{1+n}{n}}ds < \infty$.*

Proof Note that there is a positive number p with

$$|g(x)| = |x^{1/n}| = \Big(|x^{1/n}|^{(n+1)}\Big)^{\frac{1}{n+1}} = p\bigg(\int_0^x s^{1/n}ds\bigg)^{\frac{1}{n+1}}.$$

Hence $f(r) = p(r)^{\frac{1}{n+1}}$. We then have

$$V_3'(t) \leq p(V_3(t))^{\frac{1}{n+1}}|a'(t)+ka(t)|$$

and

$$\frac{1}{p}\int_{V_3(0)}^{V_3(t)} \frac{ds}{s^{\frac{1}{n+1}}} = \frac{1}{p}s^{1-\frac{1}{n+1}}\Big|_{V_3(0)}^{V_3(t)}$$

so

$$\frac{1}{p}V_3(t)^{\frac{n}{n+1}} \leq \frac{1}{p}V_3(0)^{\frac{n}{n+1}} + \int_0^t |a'(s)+ka(s)|ds.$$

Hence, $V_3(t)$ is bounded so $x(t)$ is bounded and $x(t)[a'(t)+ka(t)] \in L^1[0,\infty)$. But

$$V_3'(t) \leq |g(x)||a'(t)+ka(t)| - kxg(x)$$

with $xg(x) = xx^{1/n} = x^{1+\frac{1}{n}}$ so $\int_0^\infty |x(s)|^{\frac{1+n}{n}}ds < \infty$.

Notice that $V_3'(t) \leq -\beta g^2(x) + \alpha(a'(t)+ka(t))^2$ with $a'+ka \in L^2$ would yield $\int_0^\infty x^{2/n}(s)ds < \infty$, an entirely different property. Suppose now that $n > 2$ and that $C(t,t) \geq \alpha > 0$ so that both of our integral relations hold. Note that if $|x(t)| \geq 1$ then $|x(t)| \leq |x|^{1+\frac{1}{n}}$. If $|x(t)| < 1$ then $|x(t)| \leq |x(t)|^{2/n}$. Hence, we conclude that $\int_0^\infty |x(s)|ds < \infty$.

4 The Linear Case

If $g(x) = x$ then (7) becomes

$$x'(t) = a'(t) + ka(t) - [k + C(t,t)]x(t) - \int_0^t D(t,s)x(s)ds \tag{17}$$

and Becker's [1] resolvent equation is

$$Z_t(t,s) = -[k + C(t,t)]Z(t,s) - \int_s^t D(t,u)Z(u,s)du,\ Z(s,s) = 1, \tag{18}$$

(where $Z_t = \frac{\partial Z}{\partial t}$) with variation of parameters formula

$$x(t) = Z(t,0)x(0) + \int_0^t Z(t,s)[a'(s) + ka(s)]ds. \tag{19}$$

The Grossman-Miller [8] resolvent equation is

$$H_s(t,s) = H(t,s)[k + C(s,s)] + \int_s^t H(t,u)D(u,s)du,\ H(t,t) = 1, \tag{20}$$

and it is true that

$$H(t,s) = Z(t,s). \tag{21}$$

With D convex, a Liapunov functional for (18) is

$$V_4(t) = Z^2(t,s) + \int_s^t D_u(t,u)\left(\int_u^t Z(v,s)dv\right)^2 du + D(t,s)\left(\int_s^t Z(v,s)dv\right)^2. \tag{22}$$

Theorem 4.1 *If D is convex and $k > 0$ then the derivative of V_4 along a solution of (18) satisfies*

$$V_4'(t) \le -2[k + C(t,t)]Z^2(t,s), \text{ and } \sup_{0 \le s \le t < \infty} \int_s^t Z^2(u,s)du < \infty. \tag{24}$$

Proof We have

$$V_4'(t) \le -2[k + C(t,t)]Z^2(t,s) - 2Z(t,s)\int_s^t D(t,u)Z(u,s)du$$
$$+ 2Z(t,s)D(t,s)\int_s^t Z(v,s)dv + 2Z(t,s)\int_s^t D_u(t,u)\int_u^t Z(v,s)dvdu.$$

An integration of the last term by parts yields

$$2Z(t,s)\left[D(t,u)\int_u^t Z(v,s)dv\Big|_s^t + \int_s^t D(t,u)Z(u,s)du\right]$$
$$= 2Z(t,s)\left[-D(t,s)\int_s^t Z(v,s)dv + \int_s^t D(t,u)Z(u,s)du\right].$$

Cancellation of terms yields the required conclusion.

We then see that

$$Z^2(t,s) \leq V_4(t) \leq V_4(s) - 2k\int_s^t Z^2(u,s)du$$

with $Z^2(s,s) = 1$ yielding

$$Z^2(t,s) + 2k\int_s^t Z^2(u,s)du \leq 1. \tag{25}$$

This is a significant difference from the integral equation resolvent which requires $\int_s^t C^2(u,s)du$ bounded in order to get the parallel conclusion for the resolvent. Notice that $\int_0^t Z^2(u,0)du \leq 1/(2k)$; as $k \to \infty$, the integral tends to zero.

It is most direct to obtain $x \in L^2[0,\infty)$ in the linear case from (1) with the Liapunov functional

$$V_1(t) = \int_0^t C_s(t,s)\left(\int_s^t x(u)du\right)ds + C(t,0)\left(\int_0^t x(u)du\right)^2,$$

yielding

$$V_1'(t) \leq -x^2(t) + a^2(t).$$

We are coming to one of our central issues. From $a \in L^2$ we obtain $x \in L^2$ and, hence, from (29) we have by default that

$$\int_0^t R(t,s)a(s)ds \in L^2 \text{ and } x - a \in L^2.$$

However, we have no independent property of R which can be used without $a(t)$. We seek integral properties on R alone and the following is a typical way in which we would use them. Recall that in Section 1 we found that for C convex, then

$$\int_s^t R^2(u,s)ds \leq \int_s^t C^2(u,s)du \leq \Gamma \leq +\infty.$$

We just noted that $a \in L^2$ yields $x - a \in L^2$ by default. But $\Gamma < \infty$ yields $x - a \in L^2$ by direct computation, not by default, and that is such a desirable property in other contexts.

Proposition 4.1 *If $\Gamma < \infty$ then $a \in L^1[0,\infty)$ implies $x - a \in L^2[0,\infty)$.*

We will give a proof of a parallel result below, but it is sketched as follows.

$$(x(t) - a(t))^2 = \left(-\int_0^t R(t,s)a(s)ds\right)^2$$

so integration, followed by the Schwarz inequality and interchange of the order of integration will yield the result.

Our focus here is on the case of $\Gamma = +\infty$ and we attempt to obtain an integrability property of a resolvent. The first step is to note that (25) did not require $\Gamma < \infty$.

Proposition 4.2 *If (25) holds, then $a' + ka \in L^1[0,\infty)$ implies $x \in L^2[0,\infty)$ and $x - Z(t,0)x(0) \in L^2[0,\infty)$.*

Proof Let $a'(t) + ka(t) =: p(t)$ and from (19) we have

$$(1/2)x^2(t) \le Z^2(t,0)x^2(0) + \left(\int_0^t Z(t,s)p(s)ds\right)^2.$$

The last term is in L^1, not by default, but by the nonconvolution extension of the classical theorem that the convolution of an L^1-function with an L^2-function is an L^2-function. Here are the details. We have

$$\begin{aligned}(1/2)\int_0^t x^2(u)du &\le \int_0^t Z^2(u,0)x^2(0)du + \int_0^t \left(\int_0^u Z(u,s)p(s)ds\right)^2 du \\ &\le \int_0^t Z^2(u,0)x^2(0)du + \int_0^t \int_0^u |p(s)|ds \int_0^u Z^2(u,s)|p(s)|dsdu \\ &\le \int_0^t Z^2(u,0)x^2(0)du + \int_0^\infty |p(s)|ds \int_0^t \int_s^t Z^2(u,s)du|p(s)|ds \\ &\le \int_0^t Z^2(u,0)x^2(0)du + \left(\int_0^\infty |p(s)|ds\right)^2 (1/2k).\end{aligned}$$

We will now see how this applies to $R(t,s)$.

5 Relations Between Resolvents

If we begin with C convex and

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds, \tag{26}$$

we have the resolvent equation

$$R(t,s) = C(t,s) - \int_s^t C(t,u)R(u,s)du \tag{27}$$

and the variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds. \tag{28}$$

For (26) there is the Liapunov functional

$$V_1(t) = \int_0^t C_s(t,s)\left(\int_s^t x(u)du\right)^2 ds + C(t,0)\left(\int_0^t x(u)du\right)^2 \tag{29}$$

and a calculation given in the appendix yields

$$V_1'(t) \le -x^2(t) + a^2(t) \tag{30}$$

and

$$\int_0^t x^2(s)ds \le \int_0^t a^2(s)ds. \tag{31}$$

In a parallel manner we have a Liapunov functional for the resolvent equation given by

$$V_2(t) = \int_s^t C_2(t,u)\left(\int_u^t R(v,s)dv\right)^2 du + C(t,s)\int_s^t R(u,s)du\Big)^2 \tag{32}$$

and a calculation will yield

$$V_2'(t) \le -R^2(t,s) + C^2(t,s) \tag{33}$$

with

$$\int_s^t R^2(u,s)du \le \int_s^t C^2(u,s)du. \tag{34}$$

We explored consequences of these relations in [6] for the case

$$\sup_{0\le s\le t<\infty} \int_s^t C^2(u,s)du < \infty. \tag{35}$$

Here, we look at the case where

$$\sup_{0\le s\le t<\infty} \int_s^t C^2(u,s)du = \infty \tag{36}$$

so that V_2 yields nothing about R. We find a substitute for

$$\sup_{0\le s\le t<\infty} \int_s^t R^2(u,s)du < \infty \tag{37}$$

when (36) holds.

Theorem 5.1 *If D is defined in (6), D convex, and if $\frac{d}{ds}C(s,s)$ is continuous, then*

$$R(t,s) = Z_s(t,s) - kZ(t,s). \tag{38}$$

Proof From (17), (18), and (24) we see that for (1)

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds$$

and

$$\begin{aligned}
x(t) &= Z(t,0)a(0) + \int_0^t Z(t,s)[a'(s)+ka(s)]ds \\
&= Z(t,0)a(0) + Z(t,s)a(s)\Big|_0^t - \int_0^t Z_s(t,s)a(s)ds + k\int_0^t Z(t,s)a(s)ds \\
&= Z(t,0)a(0) + a(t) - Z(t,0)a(0) - \int_0^t [Z_s(t,s)+kZ(t,s)]a(s)ds \\
&= a(t) - \int_0^t [Z_s(t,s) - kZ(t,s)]a(s)ds.
\end{aligned}$$

This means that for any $a(t)$ with $a'(t)$ continuous then

$$\int_0^t R(t,s)a(s)ds = \int_0^t [Z_s(t,s) - kZ(t,s)]a(s)ds. \tag{39}$$

Looking back at the Grossman-Miller resolvent (20) and noting that $H(t,s) = Z(t,s)$ we see that if $C(s,s)$ has a continuous derivative, then $H_{ss} = Z_{ss}$ is continuous. We should also note from (27) that R_s is continuous. Thus, for any fixed t we can pick $a(s) = Z_s(t,s) - kZ(t,s) - R(t,s)$ and have from (39) with t fixed that

$$\int_0^t [Z_s(t,s) - kZ(t,s) - R(t,s)]^2 ds = 0. \tag{40}$$

Thus, the integrand is identically zero and (38) holds. This completes the proof.

The variation of parameters formula for (1) now becomes

$$x(t) = a(t) - \int_0^t [Z_s(t,s) - Z(t,s)]a(s)ds. \tag{41}$$

We have independent properties of Z, as well as Z_s through (20) and through integration by parts.

6 Appendix

In Section 1, we mentioned that our Liapunov functional

$$V_1(t) = \int_0^t C_s(t,s)\left(\int_s^t g(x(u))du\right)^2 ds + C(t,0)\left(\int_0^t g(x(u))du\right)^2$$

has a derivative along a solution of (1) satisfying

$$V_1'(t) \le 2g(x)[a(t) - x(t)].$$

Owing to the absence of a chain rule, that differentiation is not simple so we want to give the details. Using Leibnitz's rule we have

$$\begin{aligned} V_1'(t) = &\int_0^t C_{st}(t,s)\left(\int_s^t g(x(u))du\right)^2 ds + 2g(x)\int_0^t C_s(t,s)\int_s^t g(x(u))duds \\ &+ C_t(t,0)\left(\int_0^t g(x(s))ds\right)^2 + 2g(x)C(t,0)\int_0^t g(x(s))ds. \end{aligned}$$

We now integrate the third-to-last term by parts to obtain

$$\begin{aligned} &2g(x)\left[C(t,s)\int_s^t g(x(u))du\Big|_0^t + \int_0^t C(t,s)g(x(s))ds\right] \\ &= 2g(x)\left[-C(t,0)\int_0^t g(x(u))du + \int_0^t C(t,s)g(x(s))ds\right]. \end{aligned}$$

Cancel terms, use the sign conditions, and use (1) in the last step of the process to unite the Liapunov functional and the equation obtaining

$$\begin{aligned} V_1'(t) = &\int_0^t C_{st}(t,s)\left(\int_s^t g(x(u))du\right)^2 ds + C_t(t,0)\left(\int_0^t g(x(s))ds\right)^2 \\ &+ 2g(x)[a(t) - x(t)] \le 2g(x)[a(t) - x(t)]. \end{aligned}$$

References

[1] Becker, Leigh C. Principal matrix solutions and variation of parameters for a Volterra integro-differential equation and its adjoint. *E. J. Qualitative Theory of Diff. Eq.* **14** (2006) 1–22.

[2] Burton, T. A. Examples of Lyapunov functionals for non-differentiated equations. In: *Proc. First World Congress of Nonlinear analysts, 1992 (V. Lakshmikantham, ed.).* Walter de Gruyter, New York, 1996, 1203–1214.

[3] Burton, T. A. Boundedness and periodicity in integral and integro-differential equations. *Diff. Eq. Dynamical Systems* **1** (1993) 161–172.

[4] Burton, T. A. Scalar nonlinear integral equations. *Tatra Mt. Math. Publ.* **38** (2007) 41–56.

[5] Burton, T. A. *Liapunov Functionals for Integral Equations.* Trafford, Victoria, B. C., Canada, 2008. (www.trafford.com/08-1365)

[6] Burton, T. A. and Dwiggins, D. P. Resolvents, integral equations, limit sets. Mathematica Bohemica, to appear.

[7] Burton, T. A. and Haddock, John R. Qualitative properties of solutions of integral equations. *Nonlinear Analysis* **71** (2009) 5712–5723.

[8] Grossman, S. I. and Miller, R. K. Perturbation theory for Volterra integrodifferential systems. *J. Differential Equations* **8** (1970) 457–474.

[9] Levin, J. J. The asymptotic behavior of the solution of a Volterra equation. *Proc. Amer. Math. Soc.* **14** (1963) 534–541.

[10] Levin, J. J. The qualitative behavior of a nonlinear Volterra equation. *Proc. Amer. Math. Soc.* **16** (1965) 711–718.

[11] Londen, Stig-Olof. On the solutions of a nonlinear Volterra equation. *J. Math. Anal. Appl.* **39** (1972) 564-573.

[12] Miller, Richard K. *Nonlinear Volterra Integral Equations* Benjamin, New York, 1971.

[13] Volterra, V. Sur la théorie mathématique des phénomès héréditaires. *J. Math. Pur. Appl.* **7** (1928) 249–298.

[14] Zhang, Bo. Boundedness and global attractivity of solutions for a system of nonlinear integral equations. *Cubo: A Mathematical Journal* **11** (2009) 41–53.