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# Multi-Point Boundary Value Problems on Time Scales

İsmail Yaslan*

*Department of Mathematics, Pamukkale University 20070 Denizli, Turkey*

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**Abstract:** In this paper, we are interested in the existence of at least one,two and three positive solutions of a nonlinear second-order $m$-point boundary value problem on time scales by using fixed point theorems in cones. As an application, some examples are included to demonstrate the main results.

**Keywords:** *boundary value problems; cone; fixed point theorems; positive solutions; time scales.*

**Mathematics Subject Classification (2000):** 34B18, 39A10.

## 1 Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [7, 8]. Motivated by the study of Il'in and Moiseev [7, 8], Gupta [5] studied certain three-point boundary value problems for nonlinear ordinary differential equations. For the existence problems of positive solutions of multi-point boundary value problems on time scales, some authors have obtained many results in recent years, see [6, 9, 10, 12, 13, 14, 15, 16, 18] and the references therein.

Motivated by [17], in this paper, we are interested in the existence of multiple positive solutions of the following $m$-point boundary value problem (BVP)

$$\begin{cases} u^{\Delta\nabla}(t) + h(t)f(t, u(t)) = 0, \ \ t \in [t_1, t_m] \subset \mathbb{T}, \\ u^{\Delta}(t_m) = 0, \quad \alpha u(t_1) - \beta u^{\Delta}(t_1) = \sum\limits_{i=2}^{m-1} u^{\Delta}(t_i), \quad m \geq 3, \end{cases} \tag{1}$$

where $\mathbb{T}$ is a time scale, $0 \leq t_1 < \ldots < t_{m-1} < t_m$, $\alpha > 0$ and $\beta \geq 0$ are given constants. Some basic definitions and theorems on time scales can be found in the books [2, 3].

* Corresponding author: mailto:iyaslan@pau.edu.tr

 

The rest of paper is arranged as follows. In Section 2, we give several lemmas to prove the main results in this paper. In Section 3, we first establish the existence results of solutions of the BVP (1) as a result of Schauder fixed-point theorem. Second, we use Krasnosel'skii fixed-point theorem to show the existence of a positive solution for the BVP (1). Third, we apply the Avery-Henderson fixed-point theorem to prove the existence of at least two positive solutions to the BVP (1). Finally, we establish criteria for the existence of at least three positive solutions of the BVP (1) by using Legget-Williams fixed-point theorem. In Section 4, we give two examples to illustrate our results.

## 2 Preliminaries

We now state and prove several lemmas which are needed later. These lemmas are based on the linear BVP

$$\begin{cases} u^{\Delta\nabla}(t) + y(t) = 0, \ \ t \in [t_1, t_m] \subset \mathbb{T}, \\ u^{\Delta}(t_m) = 0, \quad \alpha u(t_1) - \beta u^{\Delta}(t_1) = \sum\limits_{i=2}^{m-1} u^{\Delta}(t_i), \quad m \geq 3. \end{cases} \tag{2}$$

**Lemma 2.1** *Let $\alpha \neq 0$ and $y \in C_{ld}[t_1, t_m]$. Then the BVP (2) has the unique solution*

$$u(t) = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) y(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s + \int_{t}^{t_m} (t - s) y(s) \nabla s. \tag{3}$$

***Proof*** From $u^{\Delta\nabla}(t) + y(t) = 0$, we have

$$u(t) = u(t_m) + u^{\Delta}(t_m)(t_m - t) + \int_{t}^{t_m} (t - s) y(s) \nabla s.$$

By using the boundary conditions, we get

$$\alpha u(t_m) + \alpha \int_{t_1}^{t_m} (t_1 - s) y(s) \nabla s - \beta \int_{t_1}^{t_m} y(s) \nabla s = \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s.$$

Since

$$u(t_m) = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) y(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s,$$

we obtain

$$u(t) = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) y(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s + \int_{t}^{t_m} (t - s) y(s) \nabla s. \ \square$$

**Lemma 2.2** *If $\alpha > 0$, $\beta \geq 0$ and $y \in C_{ld}([t_1, t_m], [0, \infty))$, then the unique solution $u$ of the BVP (2) given in (3) satisfies*

$$u(t) \geq 0, \ \ t \in [t_1, t_m] \subset \mathbb{T}.$$

***Proof*** Since $u(t)$ is increasing on $[t_1, t_m]$, we know that if $u(t_1) \geq 0$, then $u(t) \geq 0$ for $t \in [t_1, t_m]$.

$$\begin{aligned} u(t_1) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) y(s)\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s)\nabla s + \int_{t_1}^{t_m} (t_1 - s) y(s)\nabla s \\ &= \frac{\beta}{\alpha} \int_{t_1}^{t_m} y(s)\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s)\nabla s \\ &\geq 0. \end{aligned}$$

Hence the result holds. □

**Lemma 2.3** *If $\alpha > 0$, $\beta \geq 0$ and $y \in C_{ld}([t_1, t_m], [0, \infty))$, then the unique solution to the BVP (2) satisfies*

$$u(t) \geq \frac{t - t_1}{t_m - t_1} \|u\|, \quad t \in [t_1, t_m] \subset \mathbb{T}, \tag{4}$$

*where* $\|u\| = \sup\limits_{t \in [t_1, t_m]} |u(t)|$.

***Proof*** From the fact that $u(t)$ is increasing on $[t_1, t_m]$, we have $\|u\| = \sup\limits_{t \in [t_1, t_m]} |u(t)| = u(t_m)$. Let

$$g(t) = u(t) - \frac{t - t_1}{t_m - t_1} \|u\|, \quad t \in [t_1, t_m] \subset \mathbb{T}. \tag{5}$$

Since $g^{\Delta\nabla}(t) = u^{\Delta\nabla}(t) = -y(t) \leq 0$, we know that the graph of $g$ is concave on $[t_1, t_m] \subset \mathbb{T}$. We get

$$g(t_1) = u(t_1) \geq 0$$

and

$$g(t_m) = 0.$$

From the concavity of $g$,

$$g(t) \geq 0 \text{ for } t \in [t_1, t_m] \subset \mathbb{T}. \tag{6}$$

From (5) and (6), we obtain

$$u(t) \geq \frac{t - t_1}{t_m - t_1} \|u\| \text{ for } t \in [t_1, t_m] \subset \mathbb{T}. \quad \square$$

We assume the following hypotheses:

**(H1)** $h \in C_{ld}([t_1, t_m], [0, \infty))$ and there exists $t_0 \in [t_1, t_m]$ such that $h(t_0) > 0$.
**(H2)** $f : [t_1, t_m] \times [0, \infty) \to [0, \infty)$ is continuous such that $f(t, .) > 0$ on any subset of $\mathbb{T}$ containing $t_0$.

The solutions of the BVP (1) are the fixed points of the operator A defined by

$$\begin{aligned} Au(t) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s))\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s))\nabla s \\ &+ \int_{t}^{t_m} (t - s) h(s) f(s, u(s))\nabla s. \end{aligned}$$

## 3 Existence of Solutions

To prove the existence of at least one solution for the BVP (1), we will apply the following Schauder Fixed Point Theorem: *Let $\mathcal{B}$ be a Banach space and $\mathcal{S}$ be a nonempty bounded, convex, and closed subset of $\mathcal{B}$. Assume $A : \mathcal{B} \to \mathcal{B}$ is a completely continuous operator. If the operator $A$ leaves the set $\mathcal{S}$ invariant, i.e. if $A(\mathcal{S}) \subset \mathcal{S}$, then $A$ has at least one fixed point in $\mathcal{S}$.*

Let $\mathcal{B}$ denote the Banach space $C_{ld}[t_1, t_m]$ with the norm $\|u\| = \sup_{t\in[t_1,t_m]} |u(t)|$.

**Theorem 3.1** *Assume* (H1) and (H2) *are satisfied, $\alpha > 0$ and $\beta \geq 0$. Let there exists a number $r > 0$ such that*

$$\max_{\|u\|\leq r} |f(t, u)| \leq \frac{1}{k_1} u$$

*for $t \in [t_1, t_m]$, where*

$$k_1 = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s)\nabla s.$$

*Then the $m$-point BVP* (1) *has at least one solution $u(t)$.*

***Proof*** Let $\mathcal{S} = \{u \in \mathcal{B} : \|u\| \leq r\}$. Obviously, $\mathcal{S}$ is closed, bounded and convex subset of $\mathcal{B}$. Define $A : \mathcal{S} \to \mathcal{B}$ by

$$\begin{aligned} Au(t) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)f(s, u(s))\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s)f(s, u(s))\nabla s \\ &+ \int_{t}^{t_m} (t - s)h(s)f(s, u(s))\nabla s. \end{aligned}$$

for $t \in [t_1, t_m]$. Now, we will show that $A : \mathcal{S} \to \mathcal{S}$. If $u \in \mathcal{S}$,

$$\begin{aligned} \|Au\| &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)f(s, u(s))\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s)f(s, u(s))\nabla s \\ &\leq \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)\frac{1}{k_1}u(s)\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s)\frac{1}{k_1}u(s)\nabla s \\ &\leq \|u\| \leq r. \end{aligned}$$

for every $t \in [t_1, t_m]$. Since $\|Au\| \leq r$, we have $A(\mathcal{S}) \subset \mathcal{S}$. Further, the operator $A$ is completely continuous. Hence, $A$ has at least one fixed point in $\mathcal{S}$ by Schauder fixed point theorem. Since the solutions of problem (1) are fixed points of operator $A$, the BVP (1) has at least one solution $u(t)$. □

We will need also the following (Krasnosel'skii) fixed point theorem [15] to prove the existence of at least one positive solution for the BVP (1).

**Theorem 3.2** [4] *Let $E$ be a Banach space, and let $K \subset E$ be a cone. Assume $\Omega_1$ and $\Omega_2$ are open bounded subsets of $E$ with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

*be a completely continuous operator such that either*
*(i) $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$;*
*or*
*(ii) $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$*
*hold. Then $A$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

**Theorem 3.3** *Assume* (H1), (H2) *hold, and $\alpha > 0$, $\beta \geq 0$. In addition, let there exist numbers $0 < r < R < \infty$ such that*

$$f(s, u) \leq \frac{1}{k_1} u, \text{ if } 0 \leq u \leq r, \;\; s \in [t_1, t_m]$$

*and*

$$f(s, u) \geq \frac{t_m - t_1}{k_2(t_{m-1} - t_1)} u, \text{ if } R \leq u < \infty, \;\; s \in [t_{m-1}, t_m],$$

*where*

$$k_1 = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) \nabla s$$

*and*

$$k_2 = \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) \nabla s.$$

*Then the BVP (1) has at least one positive solution.*

***Proof*** Define the cone $P \subset \mathcal{B}$ by

$$P = \{u \in \mathcal{B} : \; u \text{ is concave}, u(t) \geq 0 \text{ and } u^{\Delta}(t_m) = 0\}. \tag{7}$$

From (H1), (H2), Lemma 2.2 and Lemma 2.3, we have $AP \subset P$. Also it is easy to obtain that $A : P \to P$ is completely continuous. If $u \in P$ with $\|u\| = r$, then we get

$$\begin{aligned} \|Au\| &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\ &\leq \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) \frac{1}{k_1} u(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) \frac{1}{k_1} u(s) \nabla s \\ &\leq \|u\|. \end{aligned}$$

Thus, we have $\|Au\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$, where $\Omega_1 := \{u \in C_{ld}([t_1, t_m], \mathbb{R}) : \|u\| < r\}$.

Let us now define

$$\Omega_2 := \{u \in C_{ld}([t_1, t_m], \mathbb{R}) : \|u\| < \frac{t_m - t_1}{t_{m-1} - t_1} R\}.$$

If $u \in P \cap \partial\Omega_2$, from (4)

$$u(t) \geq u(t_{m-1}) \geq \frac{t_{m-1} - t_1}{t_m - t_1}\|u\| = R, \;\; t \in [t_{m-1}, t_m]$$

and so

$$\begin{aligned}
\|Au\| &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\
&= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s \\
&+ \frac{1}{\alpha}[(\int_{t_2}^{t_3} h(s) f(s, u(s)) \nabla s + \ldots + \int_{t_{m-1}}^{t_m} h(s) f(s, u(s)) \nabla s) \\
&+ (\int_{t_3}^{t_4} h(s) f(s, u(s)) \nabla s + \ldots + \int_{t_{m-1}}^{t_m} h(s) f(s, u(s)) \nabla s) + \ldots \\
&+ \int_{t_{m-1}}^{t_m} h(s) f(s, u(s)) \nabla s] \\
&\geq \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s \\
&\geq \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) \frac{t_m - t_1}{k_2(t_{m-1} - t_1)} u(s) \nabla s \\
&\geq \|u\|.
\end{aligned}$$

Hence, $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. By the first part of Theorem 3.2, $A$ has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that $r \leq \|u\| \leq \frac{t_m - t_1}{t_{m-1} - t_1} R$. Therefore, the BVP (1) has at least one positive solution. □

Now, we apply the following (Avery-Henderson) fixed point theorem [1] to prove the existence of at least two positive solutions to the nonlinear $m$-point BVP (1).

**Theorem 3.4** [1] *Let $P$ be a cone in a real Banach space $E$. Set*

$$P(\phi, r) = \{u \in P : \phi(u) < r\}.$$

*If $\eta$ and $\phi$ are increasing, nonnegative continuous functionals on $P$, let $\theta$ be a nonnegative continuous functional on $P$ with $\theta(0) = 0$ such that, for some positive constants $r$ and $M$,*

$$\phi(u) \leq \theta(u) \leq \eta(u) \text{ and } \|u\| \leq M\phi(u)$$

*for all* $u \in \overline{P(\phi, r)}$*. Suppose that there exist positive numbers* $p < q < r$ *such that*

$$\theta(\lambda u) \leq \lambda\theta(u), \text{ for all } 0 \leq \lambda \leq 1 \text{ and } u \in \partial P(\theta, q).$$

*If* $A : \overline{P(\phi, r)} \to P$ *is a completely continuous operator satisfying*

(*i*) $\phi(Au) > r$ *for all* $u \in \partial P(\phi, r)$,

(*ii*) $\theta(Au) < q$ *for all* $u \in \partial P(\theta, q)$,

(*iii*) $P(\eta, p) \neq \emptyset$ *and* $\eta(Au) > p$ *for all* $u \in \partial P(\eta, p)$,

*then* $A$ *has at least two fixed points* $u_1$ *and* $u_2$ *such that*

$$p < \eta(u_1) \text{ with } \theta(u_1) < q \text{ and } q < \theta(u_2) \text{ with } \phi(u_2) < r.$$

Define the constants

$$M := \Big( \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + t_{m-1} - t_1) h(s) \nabla s \Big)^{-1} \tag{8}$$

and

$$N := \Big( \int_{t_1}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) \nabla s \Big)^{-1}. \tag{9}$$

**Theorem 3.5** *Assume* (H1), (H2) *hold and* $\alpha > 0$, $\beta \geq 0$. *Suppose there exist numbers* $0 < p < q < r$ *such that the function* $f$ *satisfies the following conditions:*

(*i*) $f(s, u) > rM$ *for* $s \in [t_{m-1}, t_m]$ *and* $u \in [r, \frac{r(t_m - t_1)}{t_{m-1} - t_1}]$,

(*ii*) $f(s, u) < qN$ *for* $s \in [t_1, t_m]$ *and* $u \in [0, \frac{q(t_m - t_1)}{t_{m-1} - t_1}]$,

(*iii*) $f(s, u) > pM$ *for* $s \in [t_{m-1}, t_m]$ *and* $u \in [\frac{p(t_{m-1} - t_1)}{t_m - t_1}, p]$

*where* $N$ *and* $M$ *are defined in* (8) *and* (9)*, respectively. Then the BVP* (1) *has at least two positive solutions* $u_1$ *and* $u_2$ *such that*

$$u_1(t_m) > p \text{ with } u_1(t_{m-1}) < q \text{ and } u_2(t_{m-1}) > q \text{ with } u_2(t_{m-1}) < r.$$

***Proof*** Define the cone $P$ as in (7). From (H1), (H2), Lemma 2.2 and Lemma 2.3, $AP \subset P$ and it is easy to obtain $A$ is completely continuous. Let the nonnegative increasing continuous functionals $\phi, \theta$ and $\eta$ be defined on the cone $P$ by

$$\phi(u) := u(t_{m-1}),\ \theta(u) := u(t_{m-1}),\ \eta(u) := u(t_m).$$

For each $u \in P$, we have

$$\phi(u) = \theta(u) \leq \eta(u)$$

and from (4) we have

$$\|u\| \leq \frac{t_m - t_1}{t_{m-1} - t_1} \phi(u). \tag{10}$$

Moreover, $\theta(0) = 0$ and for all $u \in P$, $\lambda \in [0, 1]$ we get $\theta(\lambda u) = \lambda\theta(u)$. In the following claims, we verify the remaining conditions of Theorem 3.5.

If $u \in \partial P(\phi, r)$, from (10) we have $r = u(t_{m-1}) \leq u(s) \leq \|u\| \leq \frac{r(t_m - t_1)}{t_{m-1} - t_1}$ for $s \in [t_{m-1}, t_m]$. Then using hypothesis $(i)$ and (8), we obtain

$$\begin{aligned}
\phi(Au) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\
&+ \int_{t_{m-1}}^{t_m} (t_{m-1} - s) h(s) f(s, u(s)) \nabla s \\
&= \int_{t_1}^{t_{m-1}} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} [\int_{t_2}^{t_{m-1}} h(s) f(s, u(s)) \nabla s + \dots \\
&+ \int_{t_{m-2}}^{t_{m-1}} h(s) f(s, u(s)) \nabla s + (m-2) \int_{t_{m-1}}^{t_m} h(s) f(s, u(s)) \nabla s] \\
&+ \int_{t_{m-1}}^{t_m} (\frac{\beta}{\alpha} + t_{m-1} - t_1) h(s) f(s, u(s)) \nabla s \\
&> \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + t_{m-1} - t_1) h(s) r M \nabla s \\
&= r.
\end{aligned}$$

Thus the condition $(i)$ of Theorem 3.4 holds. Next, we will show that the condition $(ii)$ of Theorem 3.4 is satisfied. If $u \in \partial P(\theta, q)$, then from (10) we have $0 \leq u(s) \leq \|u\| \leq \frac{q(t_m - t_1)}{t_{m-1} - t_1}$ for $s \in [t_1, t_m]$. Thus, from hypothesis $(ii)$ and (9) we get

$$\begin{aligned}
\theta(Au) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\
&+ \int_{t_{m-1}}^{t_m} (t_{m-1} - s) h(s) f(s, u(s)) \nabla s \\
&\leq \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} (m-2) \int_{t_1}^{t_m} h(s) f(s, u(s)) \nabla s \\
&< \int_{t_1}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) q N \nabla s \\
&= q.
\end{aligned}$$

So condition $(ii)$ of Theorem 3.4 holds. Since $0 \in P$ and $p > 0$, $P(\eta, p) \neq \emptyset$. If $u \in \partial P(\eta, p)$, from (4) we have $\frac{p(t_{m-1} - t_1)}{t_m - t_1} \leq u(t_{m-1}) \leq u(s) \leq \|u\| = p$ for $s \in [t_{m-1}, t_m]$.

Hence, we obtain

$$\begin{aligned}\eta(Au) &= \int_{t_1}^{t_m}(\frac{\beta}{\alpha}+s-t_1)h(s)f(s,u(s))\nabla s+\frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_i}^{t_m}h(s)f(s,u(s))\nabla s\\ &> \int_{t_{m-1}}^{t_m}(\frac{\beta+m-2}{\alpha}+t_{m-1}-t_1)h(s)pM\nabla s\\ &= p\end{aligned}$$

using hypothesis $(iii)$ and (8). Since all the conditions of Theorem 3.4 are satisfied, the $m$-point BVP (1) has at least two positive solutions $u_1$ and $u_2$ such that

$$u_1(t_m) > p \text{ with } u_1(t_{m-1}) < q \ \text{ and } \ u_2(t_{m-1}) > q \text{ with } u_2(t_{m-1}) < r. \ \square$$

Now, we will use the following (Legget-Williams) fixed point theorem [11] to prove the existence of at least three positive solutions to the nonlinear BVP (1).

**Theorem 3.6** [11] *Let $P$ be a cone in the real Banach space $E$. Set*

$$P_r := \{x \in P : \|x\| < r\}$$

$$P(\psi,a,b) := \{x \in P : a \le \psi(x),\ \|x\| \le b\}.$$

*Suppose $A : \overline{P_r} \to \overline{P_r}$ be a completely continuous operator and $\psi$ be a nonnegative continuous concave functional on $P$ with $\psi(u) \le \|u\|$ for all $u \in \overline{P_r}$. If there exists $0 < p < q < l \le r$ such that the following conditions hold,*

*$(i)$ $\{u \in P(\psi,q,l) : \psi(u) > q\} \ne \emptyset$ and $\psi(Au) > q$ for all $u \in P(\psi,q,l)$;*

*$(ii)$ $\|Au\| < p$ for $\|u\| \le p$;*

*$(iii)$ $\psi(Au) > q$ for $u \in P(\psi,q,r)$ with $\|Au\| > l$,*

*then $A$ has at least three fixed points $u_1, u_2$ and $u_3$ in $\overline{P_r}$ satisfying*

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \text{ with } \psi(u_3) < q.$$

**Theorem 3.7** *Assume* (H1), (H2) *hold and $\alpha > 0$, $\beta \ge 0$. Suppose that there exist constants $0 < p < q < \frac{q(t_m-t_1)}{t_{m-1}-t_1} \le r$ such that the function $f$ satisfies the following conditions:*

*$(i)$ $f(s,u) \le rN$ for $s \in [t_1,t_m]$ and $u \in [0,r]$;*

*$(ii)$ $f(s,u) > qM$ for $s \in [t_{m-1},t_m]$ and $u \in [q, \frac{q(t_m-t_1)}{t_{m-1}-t_1}]$;*

*$(iii)$ $f(s,u) < pN$ for $s \in [t_1,t_m]$ and $u \in [0,p]$.*

*Then the BVP* (1) *has at least three positive solutions $u_1, u_2$ and $u_3$ satisfying*

$$u_1(t_m) < p,\ u_2(t_{m-1}) > q,\ u_3(t_m) > p \text{ with } u_3(t_{m-1}) < q.$$

***Proof*** We will show that the conditions of Theorem 3.6 are satisfied. For this purpose we first define the nonnegative continuous concave functional $\psi : P \to [0,\infty)$ to be

$\psi(u) := u(t_{m-1})$, the cone $P$ as in (7), $M$ as in (8) and $N$ as in (9). We have $\psi(u) \leq \|u\|$ for all $u \in P$. If $u \in \overline{P_r}$, then $0 \leq u \leq r$ and from the hypothesis $(i)$, we get

$$\begin{aligned}\|Au\| &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)f(s,u(s))\nabla s + \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_i}^{t_m} h(s)f(s,u(s))\nabla s \\ &\leq \int_{t_1}^{t_m} (\frac{\beta+m-2}{\alpha} + s - t_1)h(s)rN\nabla s \\ &= r.\end{aligned}$$

Thus, we obtain $A : \overline{P_r} \to \overline{P_r}$. Similarly, if $u \in \overline{P_p}$, then the hypothesis $(iii)$ yields $f(s,u(s)) < pN$ for $s \in [t_1, t_m]$. Just as above, we have $A : \overline{P_p} \to P_p$. It follows that condition $(ii)$ of Theorem 3.6 is satisfied.

Since $\frac{q(t_m-t_1)}{t_{m-1}-t_1} \in P(\psi, q, \frac{q(t_m-t_1)}{t_{m-1}-t_1})$ and $\psi(\frac{q(t_m-t_1)}{t_{m-1}-t_1}) = \frac{q(t_m-t_1)}{t_{m-1}-t_1} > q$, $\{u \in P(\psi, q, \frac{q(t_m-t_1)}{t_{m-1}-t_1}) : \psi(u) > q\} \neq \emptyset$. For all $u \in P(\psi, q, \frac{q(t_m-t_1)}{t_{m-1}-t_1})$, we get $q \leq u(t_{m-1}) \leq u(s) \leq \|u\|$ for $s \in [t_{m-1}, t_m]$. Using the assumption $(ii)$, we obtain

$$\begin{aligned}\psi(Au) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)f(s,u(s))\nabla s + \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_i}^{t_m} h(s)f(s,u(s))\nabla s \\ &+ \int_{t_{m-1}}^{t_m} (t_{m-1} - s)h(s)f(s,u(s))\nabla s \\ &> \int_{t_{m-1}}^{t_m} (\frac{\beta+m-2}{\alpha} + t_{m-1} - t_1)h(s)qM\nabla s \\ &= q.\end{aligned}$$

Hence, the condition $(i)$ of Theorem 3.6 holds.

For the condition $(iii)$ of Theorem 3.6, we suppose that $u \in P(\psi, q, r)$ with $\|Au\| > \frac{q(t_m-t_1)}{t_{m-1}-t_1}$. Then, from (4) we obtain

$$\psi(Au) = Au(t_{m-1}) \geq \frac{t_{m-1}-t_1}{t_m-t_1}\|Au\| > q.$$

Since all conditions of the Legget-Williams fixed point theorem are satisfied, the nonlinear BVP (1) has at least three positive solutions $u_1, u_2$ and $u_3$ such that

$$u_1(t_m) < p,\ u_2(t_{m-1}) > q,\ u_3(t_m) > p \text{ with } u_3(t_{m-1}) < q. \ \square$$

Using the ideas in the proof of the above problem, we can establish the existence of an arbitrary odd number of positive solutions of (1).

**Theorem 3.8** *Assume that* (H1) *and* (H2) *are satisfied and* $\alpha > 0$, $\beta \geq 0$. *Suppose that there exist numbers*

$$0 < p_1 < q_1 < \frac{q_1(t_m-t_1)}{t_{m-1}-t_1} \leq p_2 < q_2 < \frac{q_2(t_m-t_1)}{t_{m-1}-t_1} \leq p_3 < \ldots \leq p_n, \quad n \in \mathbb{N}$$

*such that the function $f$ satisfies the following conditions:*

*(i) $f(s,u) < p_i N$ for $s \in [t_1, t_m]$ and $u \in [0, p_i]$,*

*(ii) $f(s,u) > q_i M$ for $s \in [t_{m-1}, t_m]$ and $u \in [q_i, \frac{q_i(t_m - t_1)}{t_{m-1} - t_1}]$,*

*where $N$ and $M$ are defined in (8) and (9), respectively. Then the $m$-point BVP (1) has at least $2n-1$ positive solutions.*

The proof of the theorem comes directly from induction. When $n = 1$, we obtain $A : \overline{P_{p_1}} \to P_{p_1} \subset \overline{P_{p_1}}$ from condition $(i)$, which implies that $A$ has at least one fixed point $u_1 \in \overline{P_{p_1}}$ by the Schauder fixed point theorem. When $n = 2$, by Theorem 3.7 we can obtain at least three positive solutions $u_2, u_3$ and $u_4$. Following this way, we can obtain that the operator $A$ has $2n-1$ different fixed points by induction.

**Remark 3.1** When $m = 3$, our results, i.e. Theorem 3.3, Theorem 3.5 and Theorem 3.7 reduce to Theorem 4, Theorem 5 and Theorem 6 in [12], respectively.

## 4 Examples

**Example 4.1** Let $\mathbb{T} = \{(\frac{1}{5})^n : n \in \mathbb{N}_0\} \cup \{0\}$. We consider the following boundary value problem:

$$\begin{cases} u^{\Delta\nabla}(t) + \frac{29(u+4)^{\frac{1}{20}}}{(u+4)^{\frac{1}{2}}+1} = 0, \;\; t \in [0,1] \subset \mathbb{T}, \\ u^{\Delta}(1) = 0, \quad u(0) - 2u^{\Delta}(0) = u^{\Delta}(\frac{1}{25}) + u^{\Delta}(\frac{1}{5}). \end{cases} \tag{11}$$

Taking $t_1 = 0, t_2 = \frac{1}{25}, t_3 = \frac{1}{5}, t_4 = 1 = \alpha, \beta = 2, m = 4, h(t) = 1$ and $f(t,u) = \frac{29(u+4)^{\frac{1}{20}}}{(u+4)^{\frac{1}{2}}+1}$, we investigate the solvability of this problem by means of Theorem 3.5. By (8) and (9), we obtain $M = \frac{25}{84}$ and $N = \frac{6}{29}$.

If we take $p = 10, q = 17$ and $r = 19$, then $0 < p < q < r$ and the conditions $(i) - (iii)$ of Theorem 3.5 are satisfied. Thus, the BVP (11) has at least two positive solutions $u_1$ and $u_2$ satisfying

$$u_1(1) > 10 \text{ with } u_1(\frac{1}{5}) < 17 \text{ and } u_2(\frac{1}{5}) \text{ with } u_2(\frac{1}{5}) < 19.$$

**Example 4.2** In problem (11), let $f(t,u) = \frac{2u^2}{(u+1)^2+1}$. If we take $p = 0.27, q = 1$ and $r = 8$ then $0 < p < q < \frac{q(t_m - t_1)}{t_{m-1} - t_1} \leq r$ and the conditions $(i) - (iii)$ of Theorem 3.7 are satisfied. According to Theorem 3.7, the BVP

$$\begin{cases} u^{\Delta\nabla}(t) + \frac{2u^2}{(u+1)^2+1} = 0, \;\; t \in [0,1] \subset \mathbb{T}, \\ u^{\Delta}(1) = 0, \quad u(0) - 2u^{\Delta}(0) = u^{\Delta}(\frac{1}{25}) + u^{\Delta}(\frac{1}{5}), \end{cases}$$

has at least three positive solutions $u_1$, $u_2$ and $u_3$ satisfying

$$u_1(1) < 0.27, \quad u_2(\frac{1}{5}) > 1, \;\; u_3(1) > 0.27 \text{ with } u_3(\frac{1}{5}) < 1.$$

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