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Semilinear Hyperbolic Integrodifferential Equations with Nonlocal Conditions

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Abstract: In this paper, we consider an abstract nonlocal semilinear hyperbolic integrodifferential equation in a Banach space. Using the theory of resolvent operators, we establish the existence and uniqueness of a mild solution under local Lipschitz conditions on the nonlinear maps and an integrability condition on the kernel. The existence of a classical solution of the problem considered is proved under some additional conditions on the nonlinear maps.

Keywords: *hyperbolic problem, integrodifferential equation, nonlocal Cauchy problem, mild and classical solutions.*

Mathematics Subject Classification (2000): 34G20, 35L90, 34A12.

1 Introduction

In the present paper, we study the following semilinear integrodifferential equation with a nonlocal Cauchy problem:

$$u'(t) = A[u(t) + \int_{t_0}^{t} F(t-s)u(s)ds] + f(t,u(t)) + \int_{t_0}^{t} k(t-s)h(s,u(s))ds, \tag{1.1}$$

$$u(t_0) + g(t_1, \cdots, t_n, u(t_1), \cdots, u(t_n)) = u_0 \in E,\ t \in [t_0, T], \tag{1.2}$$

where $t_0 < t_1 < t_2 < \ldots < t_n \leq T$, $(n \in \mathbb{N})$, $A : D(A) := D \subset E \to E$ is a linear operator and generates the strongly continuous semigroup $S(t)$, the nonlinear maps f, h are defined as $f, h : [t_0, T] \times E \to E$, $g : I_T^n \times E^n \to E$, $F(t) \in B(E)$, $t \in I_T$,

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$F(t) : Y \to Y$ where Y is the Banach space $D(A)$, endowed with the graph norm, and the kernel k is defined on $[t_0, T]$ to $\mathbb{R}$.

We consider the following semilinear equation

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in [t_0, T], \tag{1.3}$$
$$u(t_0) = u_0, \tag{1.4}$$

in a Banach space E where $A : D(A) := D \subset E \to E$ is a linear operator and generates a strongly continuous semigroup $\{S(t) : t \geq 0\}$ and $f : [t_0, T] \times E \to E,\ u_0 \in E$ are given. Problem (1.3)-(1.4) is referred to as an abstract initial value problem, or a Cauchy problem. For applications to certain physical problems many researchers, for instance, Byszewski [5], Byszewski and Lakshmikantham [12], Jackson [15] and references therein, have considered the study of the existence and uniqueness of a mild solution and a classical solution to the following nonlocal Cauchy problem

$$u'(t) = Au(t) + f(t, u(t)), \tag{1.5}$$
$$u(t_0) + g(t_1, t_2, \ldots, t_n, u(t_1), u(t_2), \cdots, u(t_n)) = u_0,\ t \in [t_0, T], \tag{1.6}$$

where $t_0 < t_1 < t_2 < \ldots < t_n \leq T, (n \in \mathbb{N}),\ A$ is the generator of a C_0 semigroup $\{S(t) : t \geq 0\}$ on a Banach space E, $f : [t_0, T] \times E \to E$, and $g : [t_0, T]^n \times E^n \to E$ are the given functions. A possible example for a function g is

$$g(t_1, t_2, \ldots, t_n, u(t_1), u(t_2), \cdots, u(t_n)) = \sum_{i=1}^{n} c_i u(t_i). \tag{1.7}$$

The main advantage to use a nonlocal condition (1.6) is that it may be applied to a physical problem with a better effect than the classical condition (1.4) as (1.6) is generally more practical for the physical measurements as compared to the classical condition (1.4).

Recently Lin and Liu [16] have dealt with the following semilinear integrodifferential equation

$$u'(t) = A[u(t) + \int_0^t F(t-s)u(s))ds] + f(t, u(t)), \quad t \in [0, T], \tag{1.8}$$
$$u(0) + g(t_1, \cdots, t_n, u(t_1), \cdots, u(t_n)) = u_0, \tag{1.9}$$

in a Banach space E with A being the generator of a strongly continuous semigroup and $F(t)$ being a bounded linear operator for $t \in [0, T]$, by generalizing the results of (1.5)-(1.6). We note that the method used to study (1.5)-(1.6) is to first establish the existence of a mild solution using a fixed point theorem when f satisfies a Lipschitz condition in the second argument, where a mild solution is defined to be a solution of the following integral equation:

$$\begin{aligned} u(t) &= S(t - t_0)[u_0 - g(t_1, t_2, \ldots, t_n, u(t_1), u(t_2), \cdots u(t_n))] \\ &\quad + \int_{t_0}^t S(t - s) f(s, u(s)) ds, \quad t_0 \leq t \leq T, \end{aligned} \tag{1.10}$$

with $S(t)$ being the semigroup generated by A. Then a mild solution is shown to be a classic solution if $f \in C^1([t_0, T] \times E,\ E)$.

Similar approach has been used to study (1.8)–(1.9) by Lin and Liu [16] who have shown the existence and uniqueness of a mild solution by showing the existence and uniqueness of a solution of the following integral equation (generally known as variation of constants formula):

$$\begin{aligned} u(t) &= R(t)[u_0 - g(t_1, t_2, \dots, t_n, u(t_1), u(t_2), \cdots u(t_n))] \\ &\quad + \int_0^t R(t-s) f(s, u(s)) ds, \quad 0 \le t \le T, \end{aligned} \tag{1.11}$$

where the semigroup $S(t)$ is replaced by the resolvent operator $R(t)$, the counterpart of the semigroup $S(t)$ for the integrodifferential equations. Then a mild solution is shown to be a classical solution if $f \in C^1([0,T] \times E,\ E)$.

For the initial works on existence, uniqueness and stability of various types of solutions of different kinds of differential equations, we refer to [6]–[10] and the references cited in these papers.

Our aim is to use the properties of the resolvent operator $R(t)$ studied in [13]–[17] and the techniques of Pazy [18] and Byszewski [5] for proving the existence, uniqueness, representation of solutions by variation of constants formula.

We first prove the existence and uniqueness of a mild solution to (1.1), using the fixed point argument under a Lipschitz condition on the nonlinear maps and an integrability condition on the kernel k. Where by a mild solution to (1.1) we mean a function $u \in C(I_T, E)$ satisfying the following integral equation

$$\begin{aligned} u(t) &= R(t-t_0)[u_0 - g(t_1 \dots t_n, u(t_1), \cdots, u(t_n)] + \int_{t_0}^t R(t-s)[f(s, u(s))ds \\ &\quad + \int_{t_0}^s k(s-\tau) h(\tau, u(\tau)) d\tau] ds, \quad t \in [t_0, T], \end{aligned} \tag{1.12}$$

where the semigroup $S(t)$ in (1.12) is replaced by the resolvent operator $R(t)$ used in (1.11). Then a mild solution is shown to be a classical solution under certain differentiability condition on the nonlinear maps.

The organization of this paper is as follows. In Section 2, we give some basic results, assumptions on the resolvent operator $R(t)$ and on the variation of parameters formula. Then in Section 3 we will study the nonlocal Cauchy problem (1.1) using the results given in Section 2.

2 Preliminaries and Assumptions

In this section we give some basic definitions, notations and results. Let E be a Banach space with the norm $\|.\|$ and let $t_0 < T \le \infty$, and throughout the paper we denote $[t_0, T]$ by I_T . We will use in this paper the following Banach spaces of functions (endowed with their usual norms):

- $C(I_T; E)$: the space of all continuous functions $u : I_T \to E$.
- $C^n(I_T; E)$: the space of all n times continuously differentiable functions $u : I_T \to E$.
- $L^p(I_T; E)$: the space of all measurable functions $u : I_T \to E$. such that $\|u(.)\| \in L^p(I_T); 1 \le p < \infty$.

In the following, for a linear operator A on a Banach space E, we denote by Y the Banach space $D(A)$ endowed with the graph norm. By $L(E,F)$, we denote the set of all linear operators from E to F. By $B(E)$, we denote the set of all bounded linear operators from E to E itself.

We will make the following assumptions used in [16], [14] and [17] :

(D1) A generates a strongly continuous semigroup in E,

(D2) $F(t) \in B(E),\ t \in I_T,\ F(t) : Y \to Y$ and for $u : I_T \to Y$ continuous, $AF(.)u(.) \in L^1(I_T, E)$. For $u \in E,\ F'(t)u$ is continuous in $t \in I_T$.

Now, we define resolvent operator for (1.1) as follows.

Definition 2.1 (see [17]) R(.) is a resolvent operator of (1.1) with $f,\ g,\ h \equiv 0$ if $R(t) \in B(E)$ for $t \in I_T$ and satisfies

1. $R(0) = I$ (the identity operator on E),
2. for all $u \in E,\ R(t)u$ is continuous for $t \in I_T$,
3. $R(t) \in B(E), t \in I_T$. For $y \in Y, R(.)y \in C^1(I_T, E) \cap C(I_T, Y)$ and

$$\begin{aligned} \frac{d}{dt}R(t-t_0)y &= A[R(t-t_0)y + \int_{t_0}^{t} F(t-s)R(s)yds] \\ &= R(t-t_0)Ay + \int_{t_0}^{t} R(t-s)AF(s)yds,\ t \in I_T. \end{aligned} \tag{2.1}$$

Definition 2.2 $u(., u_0) \in C(I_T, E)$ is a mild solution of (1.1) if it satisfies

$$\begin{aligned} u(t) &= R(t-t_0)[u_0 - g(t_1 \dots t_n, u(t_1), \cdots, u(t_n)] + \int_{t_0}^{t} R(t-s)[f(s, u(s))ds \\ &\quad + \int_{t_0}^{s} k(s-\tau)h(\tau, u(\tau))d\tau]ds, \quad t \in [0, T]. \end{aligned} \tag{2.2}$$

Definition 2.3 A classical solution $u(., u_0)$ of (1.1) is a function $u \in C(I_T, Y) \cap C^1(I_T, E)$ which satisfies (1.1) on I_T.

Now we state here some results about the existence and uniqueness of the resolvent operators, already proved in [16] and [17].

Theorem 2.1 ([16]) *Let* (D1) *and* (D2) *be satisfied. Then* (1.1) *with* $f, g, h \equiv 0$ *has a unique resolvent operator.*

We also state a result about the classical solution to the (1.1) for the particular case, i.e. $f(t, u) \equiv f(t)$.

Theorem 2.2 ([17]) *Let assumptions* (D1) *and* (D2) *be satisfied and assume that* $f(t, u) \equiv f(t),\ g, h \equiv 0,\ u_0 \in D,$ *and* $f \in C^1(I_T, E)$. *Then* (1.1) *has a unique classical solution.*

Finally, we state a theorem about the variation of constants formula for (1.1).

Theorem 2.3 ([13],[17]) *Let $f \in C(I_T,\ E)$ and $R(t)$ be the resolvent operator for* (1.1) *with $g, h \equiv 0$. If u is a classical solution of* (1.1) *with $g, h \equiv 0$, then it satisfies the following integral equation:*

$$u(t) = R(t - t_0)u(t_0) + \int_{t_0}^{t} R(t - s)f(s)ds, \quad t \in I_T. \tag{2.3}$$

This is also known as variation of constant formula for (1.1).

3 Main Results

In this section we give the sufficient conditions for the existence and uniqueness of solutions to (1.1). We first prove the local existence and uniqueness of mild solution to (1.1) under the assumptions (D1)-(D2), $f(t, u)$, $h(t, u)$ are continuous in t and satisfy the certain local Lipschitz condition in u with Lipschitz constants depending on t and $\|u\|_E$ and $k \in L^p(I_T)$, $1 < p < \infty$. Finally, we show that (1.1) has a classical solution provided f and g are continuously differentiable from $I_T \times E \to E$.

We have the following result for a mild solution of (1.1).

Theorem 3.1 *Let* (D1) *and* (D2) *hold. Let $f, g : I_T \times E \to E$ be continuous in t on I_T and satisfy the following conditions.*

(H1) *There exists a constant $L_1 > 0$ such that*

$$\|f(t, u) - f(t, v)\| \le L_1\|u - v\|,\ u, v \in E.$$

(H2) *For almost every $t \in I_T$ and $u, v \in E$ there exists a nonnegative function $L_2 \in L^P(I_T), 1 < P < \infty$ such that*

$$\|h(t, u) - h(t, v)\| \le L_2\|u - v\|.$$

(H3) *$t_0 < t_1 < t_2 < \cdots < t_n = T, (n \in \mathbb{N})$ and $g : I_T^n \times E^n \to E$ and $\exists\ G$, a constant such that*

$$\begin{aligned}\|g(t_1, \cdots, t_n, u(t_1), \cdots, u(t_n)) - g(t_1, \cdots, t_n, v(t_1), \cdots, v(t_n))\| \\ \le G\|u - v\|_{C(I_T,E)}.\end{aligned}$$

(H4) *The real valued map k is in $L^q(0, T)$, where $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(H5) *The constants M and M_0 are defined as:*

$$\begin{aligned} M &= \max_{\tau \in I_T} \|R(\tau)\|, \\ M_0 &= M[L_1 + \|k\|_{L^q(I_T)}\|L_2\|_{L^p(I_T)}] \end{aligned} \tag{3.1}$$

and satisfy the following inequality

$$MG + (T - t_0)M_0 < 1. \tag{3.2}$$

Then for every $u_0 \in E$ the nonlocal semilinear problem (1.1) *has a unique mild solution $u \in C(I_T, E)$, Moreover, the mild solution depends continuously on initial data on I_T.*

Proof We fixed $u_0 \in E$. Define a map $X : C(I_T, E) \to C(I_T, E)$ as:

$$\begin{aligned}(Xu)(t) &= R(t-t_0)[u_0 - g(t_1, t_2, \cdots, t_n, u(t_1), u(t_2), \cdots, u(t_n))] \\ &+ \int_{t_0}^{t} R(t-s)[f(s, u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau, u(\tau))d\tau]ds. \end{aligned} \tag{3.3}$$

Now, for $u, v \in C(I_T, E)$, we have

$$\begin{aligned}\|(Xu)(t) - (Xv)(t)\| &\leq \|R(t-t_0)[g(t_1, t_2, \cdots, t_n, v(t_1), v(t_2), \cdots, v(t_n)) \\ &\quad -g(t_1, t_2, \cdots, t_n, u(t_1), u(t_2), \cdots, u(t_n))]\| \\ &\quad +\| \int_{t_0}^{t} R(t-s)[\{f(s, u(s)) - f(s, v(s))\} \\ &\quad + \int_{t_0}^{s} k(s-\tau)\{h(\tau, u(\tau)) - h(\tau, v(\tau))\}d\tau]ds\| \\ &\leq \|R(t-t_0)\|\|g(t_1, t_2, \cdots, t_n, v(t_1), v(t_2), \cdots, v(t_n)) \\ &\quad -g(t_1, t_2, \cdots, t_n, u(t_1), u(t_2), \cdots, u(t_n))\| \\ &\quad + \int_{t_0}^{t} \|R(t-s)\|[\|f(s, u(s)) - f(s, v(s))\| \\ &\quad +\| \int_{t_0}^{s} k(s-\tau)\{h(\tau, u(\tau)) - h(\tau, v(\tau))\}d\tau\|]ds \end{aligned}$$

$$\|(Xu)(t) - (Xv)(t)\|_E \leq MG + M_0(T - t_0)\|u - v\|_{C(I_T, E)}. \tag{3.4}$$

By (3.2) and the well known extension of the Banach contraction principle X has a unique fixed point $u \in C(I_T, E)$. This u satisfies (2.2) and hence it is a unique mild solution to (1.1) on I_T.

To show the continuous dependence of a mild solution u to (1.1) on the initial data, we will show the Lipschitz continuity of the map $u_0 \to u$. The arguments for this are as follows: Let v be a mild solution of (1.1) on I_T with the initial value $v(t_0) = v_0$, then

$$\|u(t) - v(t)\|_E \leq M(\|u_0 - v_0\|_E - G\|u(t) - v(t)\|_E) + M_0 \int_{t_0}^{t} \|u - v\|_{C(I_s, E)} ds. \tag{3.5}$$

Thus for $\eta \in I_t$, we have

$$\|u(\eta) - v(\eta)\|_E \leq \tilde{M}(\|u_0 - v_0\|_E) + \tilde{M}_0 \int_{t_0}^{\eta} \|u - v\|_{C(I_s, E)} ds \tag{3.6}$$

with $\tilde{M} = \frac{M}{1+MG}$, $\tilde{M}_0 = \frac{M_0}{1+MG}$. Thus taking the supremum over I_t, we have

$$\|u - v\|_{C(I_T, E)} \leq \tilde{M}(\|u_0 - v_0\|_E) + \tilde{M}_0 \int_{t_0}^{t} \|u - v\|_{C(I_s, E)} ds. \tag{3.7}$$

Applying Gronwall's inequality and taking the supremum over I_T, we get

$$\|u - v\|_{C(I_T, E)} \leq \tilde{M} \exp\{\tilde{M}_0 T\}\|u_0 - v_0\|_E. \tag{3.8}$$

The inequality (3.8) proves the uniqueness and continuous dependence of a mild solution to (1.1) on the initial data on I_T. Thus, proof of Theorem (3.1) is complete.

The proof of Theorem 3.1 can be modified to get the following result.

Corollary 3.1 *Let A, f, h and k be as in Theorem* 3.1. *Let $r \in C(I_{\tilde{T}}, E)$. Then the integral equation*

$$w(t) = r(t) + \int_{t_0}^{t} R(t-s)[f(s, w(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau, w(\tau))d\tau]ds, \ t \in I_{\tilde{T}}$$

has a unique solution in $C(I_{\tilde{T}}, E)$.

Now we show that, if we assume the conditions of differentiability on the nonlinear maps f, h, we have the regularity result, which proves the existence and uniqueness of classical solution to (1.1), given as follows.

Theorem 3.2 *Let* (D1),(D2),(H1)–(H2),(H5) *be satisfied. If $f, h : I_T \times E \to E$ are continuously differentiable from their domain into E, $g : I_T^n \times E^n \to E$ satisfies the condition* (H3) *and k is continuous on I_T satisfying* (H4)*, then the mild solution u to* (1.1) *obtained in Theorem* 3.1*, with $u_0 \in D(A)$ is a unique classical solution to* (1.1) *on I_T.*

Proof If f, h are continuously differentiable from $I_T \times E$ into E then for any compact subinterval $I_{\tilde{T}}$ of I_T, f, h are continuous in t on $I_{\tilde{T}}$ and satisfy (H1)-(H2). Therefore, (1.1) has a unique mild solution u on $I_{\tilde{T}}$ such that $u(t_0) = u_0 - g(t_1, t_2, \cdots, t_n, u(t_1), u(t_2), \cdots, u(t_n))$. To show that it is also a classical solution of (1.1), we have to show that u is continuously differentiable on $I_{\tilde{T}}$.
Let

$$B_1(t) = \frac{\partial}{\partial u} f(t, u), \tag{3.9}$$

$$B_2(t) = \frac{\partial}{\partial u} h(t, u), \tag{3.10}$$

$$\begin{aligned} r(t) &= A[R(t-t_0)u(t_0) + \int_{t_0}^{t} F(t-s)R(s)u(t_0)ds] \\ &+ R(t-t_0)f(t_0, u(t_0)) + \int_{t_0}^{t} R(t-s)k(s-t_0)h(t_0, u(t_0)) \\ &+ \int_{t_0}^{t} R(t-s)[\frac{\partial}{\partial s} f(s, u(s)) + \int_{t_0}^{s} k(s-\tau)\frac{\partial}{\partial \tau} h(\tau, u(\tau))d\tau]ds. \end{aligned} \tag{3.11}$$

Consider the integral equation

$$w(t) = r(t) + \int_{t_0}^{t} R(t-s)[B_1(s)w(s) + \int_{t_0}^{s} k(s-\tau)B_2(\tau)w(\tau)d\tau]ds. \tag{3.12}$$

Conditions assumed on f, h imply that r is continuous on $I_{\tilde{T}}$ and $B_i(t)u$ are continuous in t from $I_{\tilde{T}}$ into E and uniformly Lipschitz continuous in u. From Corollary 3.1, it follows that (3.12) has a unique mild solution w on $I_{\tilde{T}}$. Now from the assumption on f and h we have

$$f(s, u(s+\triangle)) - f(s, u(s)) = B_1(s)[u(s+\triangle) - u(s)] + \omega_1(s, \triangle), \tag{3.13}$$

$$h(s, u(s+\triangle)) - h(s, u(s)) = B_2(s)[u(s+\triangle) - u(s)] + \omega_2(s, \triangle), \tag{3.14}$$

$$f(s+\triangle, u(s+\triangle)) - f(s, u(s+\triangle)) = \frac{\partial}{\partial s} f(s, u(s+\triangle))\triangle + \omega_3(s, \triangle), \tag{3.15}$$

$$h(s+\triangle, u(s+\triangle)) - h(s, u(s+\triangle)) = \frac{\partial}{\partial s} h(s, u(s+\triangle))\triangle + \omega_4(s, \triangle), \tag{3.16}$$

where $\triangle^{-1}\ \|\omega_i(s,\triangle)\|_E \to 0$ as $\triangle\to 0$ uniformly on $I_{\tilde{T}}$, for $i = 1,2,3,4$.
Let

$$w_\triangle(t) = \frac{u(t+\triangle) - u(t)}{\triangle} - w(t),\ t \in I_{\tilde{T}}. \tag{3.17}$$

Then

$$\begin{aligned} w_\triangle(t) &= [\frac{1}{\triangle}(R(t+\triangle - t_0)u(t_0) - R(t-t_0)u(t_0)) \\ &\quad + A\{R(t-t_0)u(t_0) + \int_{t_0}^{t} F(t-s)R(s)u(t_0)ds\}] \\ &+ [\frac{1}{\triangle}\int_{t_0}^{t_0+\triangle} R(t+\triangle - s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds \\ &- R(t-t_0)f(t_0,u(t_0)) - \int_{t_0}^{t} R(t-s)k(s-t_0)h(t_0,u(t_0))ds] \\ &+ \frac{1}{\triangle}[\int_{t_0+\triangle}^{t+\triangle} R(t+\triangle - s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds \\ &- \int_{t_0}^{t} R(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds] \\ &- \int_{t_0}^{t} R(t-s)[\frac{\partial}{\partial s}f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)\frac{\partial}{\partial \tau}h(\tau,u(\tau))d\tau]ds \\ &- \int_{t_0}^{t} R(t-s)[B_1(s)w(s)) + \int_{t_0}^{s} k(s-\tau)B_2(\tau)w(\tau))d\tau]ds. \end{aligned} \tag{3.18}$$

Consider

$$\int_{t_0+\triangle}^{t+\triangle} R(t+\triangle - s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds. \tag{3.19}$$

Putting $s = \eta+\triangle$ in (3.19), and then replacing η by s, we have

$$\begin{aligned} &= \int_{t_0}^{t} R(t-\eta)[f(\eta+\triangle, u(\eta+\triangle)) + \int_{t_0}^{\eta+\triangle} k(\eta+\triangle - \tau)h(\tau,u(\tau))d\tau]d\eta. \\ &= \int_{t_0}^{t} R(t-s)[f(s+\triangle, u(s+\triangle)) \\ &\quad + \int_{t_0}^{s+\triangle} k(s+\triangle - \tau)h(\tau,u(\tau))d\tau]ds. \end{aligned} \tag{3.20}$$

Again, in the inner integral on the right of (3.20), putting $\tau = \gamma+\triangle$ and then replacing γ by τ, we get

$$\begin{aligned} &\int_{t_0}^{t} R(t-s)[f(s+\triangle, u(s+\triangle)) + \int_{t_0}^{s+\triangle} k(s+\triangle - \tau)h(\tau,u(\tau))d\tau]d\eta \\ = &\int_{t_0}^{t} R(t-s)[f(s+\triangle, u(s+\triangle)) + \int_{t_0-\triangle}^{s} k(s-\tau)h(\tau+\triangle, u(\tau+\triangle))d\tau]ds. \end{aligned}$$

The last term can be rewritten as

$$\begin{aligned} &= \int_{t_0}^{t} R(t-s)[f(s+\triangle, u(s+\triangle)) + \int_{t_0}^{s} k(s-\tau)h(\tau+\triangle, u(\tau+\triangle))d\tau]ds \\ &+ \int_{t_0}^{t}\int_{t_0-\triangle}^{t_0} R(t-s)k(s-\tau)h(\tau+\triangle, u(\tau+\triangle))d\tau ds. \end{aligned} \tag{3.21}$$

Now, using (3.21) in (3.18) we have

$$\begin{aligned} w_{\triangle}(t) &= [\frac{1}{\triangle}(R(t+\triangle-t_0))u(t_0) - R(t-t_0)u(t_0) \\ &\quad +A\{R(t-t_0)u(t_0) + \int_{t_0}^{t} F(t-s)R(s)u(t_0)ds\}] \\ &+ [\frac{1}{\triangle}\int_{t_0}^{t_0+\triangle} R(t+\triangle-s)[f(s,u(s) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds \\ &- R(t-t_0)f(t_0,u(t_0)) - \int_{t_0}^{t} R(t-s)k(s-t_0)h(t_0,u(t_0))ds] \\ &+ \frac{1}{\triangle}\{[\int_{t_0}^{t} R(t-s)[f(s+\triangle, u(s+\triangle)) \qquad (3.22) \\ &\quad + \int_{t_0}^{s} k(s-\tau)h(\tau+\triangle, u(\tau+\triangle))d\tau]ds \\ &- \int_{t_0}^{t} R(t-s)[f(s,u(s+\triangle) + \int_{t_0}^{s} k(s-\tau)h(\tau, u(\tau+\triangle))d\tau]ds] \\ &+ [\int_{t_0}^{t} R(t-s)[f(s,u(s+\triangle) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau+\triangle)d\tau]ds \\ &- \int_{t_0}^{t} R(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds] \\ &+ \int_{t_0}^{t}\int_{t_0}^{t_0-\triangle} R(t-s)k(s-\tau)h(\tau+\triangle, u(\tau+\triangle))d\tau ds\} \\ &- \int_{t_0}^{t} R(t-s)[\frac{\partial}{\partial s}f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)\frac{\partial}{\partial \tau}h(\tau,u(\tau))d\tau]ds \\ &- \int_{t_0}^{t} R(t-s)[B_1(s)w(s)) + \int_{t_0}^{s} k(s-\tau)B_2(\tau)w(\tau))d\tau]ds. \end{aligned} \tag{3.23}$$

Now, using (3.13)-(3.16) in (3.23) and readjusting the terms, we have

$$\begin{aligned} w_{\triangle}(t) &= [\frac{1}{\triangle}(R(t+\triangle-t_0))u(t_0) - R(t-t_0)u(t_0) \\ &\quad +A\{R(t-t_0)u(t_0) + \int_{t_0}^{t} F(t-s)R(s)u(t_0)ds\}] \\ &\quad +[\frac{1}{\triangle}\int_{t_0}^{t_0+\triangle} R(t+\triangle-s)[f(s,u(s+\triangle)) \end{aligned}$$

$$
\begin{aligned}
&+ \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds - R(t-t_0)f(t_0,u(t_0))] \\
&+\frac{1}{\triangle}\int_{t_0}^{t} R(t-s)[\omega_1(s,\triangle)+\omega_3(s,\triangle) \\
&+\int_{t_0}^{s} k(s-\tau)\{\omega_2(s,\triangle)+\omega_4(s,\triangle)\}d\tau]ds \\
&+\int_{t_0}^{t} R(t-s)[\{\frac{\partial}{\partial s}f(s,u(s+\triangle))-\frac{\partial}{\partial s}f(s,u(s))\} \qquad (3.24)\\
&+\int_{t_0}^{s} k(s-\tau)\{\frac{\partial}{\partial\tau}h(\tau,u(\tau+\triangle))-\frac{\partial}{\partial\tau}h(\tau,u(\tau))\}d\tau]ds \\
&-\int_{t_0}^{t} R(t-s)[\frac{1}{\triangle}\int_{t_0-\triangle}^{t_0} k(s-\tau)h(\tau+\triangle,u(\tau+\triangle))d\tau + k(s-t_0)h(t_0,u(t_0))]ds \\
&+\int_{t_0}^{t} R(t-s)[B_1(s)w_\triangle(s)+\int_{t_0}^{s} k(s-\tau)B_2(\tau)w_\triangle(\tau)d\tau]ds. \qquad (3.25)
\end{aligned}
$$

Since the norms in E of all but the term in the last line of (3.25) tend to zero as $\triangle\to 0$, we have

$$\|w_\triangle\|_{C(I_t,E)} \le \epsilon(\triangle) + D(\tilde{T})\int_{t_0}^{t}\|w_\triangle\|_{C(I_s,E)}ds, \qquad (3.26)$$

where $\epsilon(\triangle)\to 0$ as $\triangle\to 0$ and

$$D(\tilde{T}) = \max\{\|R(t-s)\|_{B(E)}[\|B_1(s)\|_{B(E)} + \|k\|_{L^p(I_T)}\|B_2(s)\|_{B(E)}] : s\in I_{\tilde{T}}\}.$$

Applying Gronwall's inequality in (3.26), we obtain

$$\|w_\triangle\|_{C(I_t,E)} \le \epsilon(\triangle)\exp\{D(\tilde{T})\tilde{T}\}. \qquad (3.27)$$

Therefore $\|w_\triangle(t)\|_{E)} \to 0$ as $\triangle\to 0$. Hence u is differentiable on $I_{\tilde{T}}$ and its derivative is w on $I_{\tilde{T}}$. Since $w\in C(I_{\tilde{T}},E),\ u\in C^1(I_{\tilde{T}},E)$. Finally, to show that u is the required classical solution of problem (1.1), assumptions on f,h and $u\in C^1(I_{\tilde{T}},E)$ imply that the maps $s\to f(s,u(s))$ and $s\to h(s,u(s))$ are continuously differentiable on $I_{\tilde{T}}$.

$$
\begin{aligned}
v(t) &= R(t-t_0)[u_0 - g(t_1,t_2,\cdots,t_n,u(t_1),u(t_2),\cdots,u(t_n))] \\
&+ \int_{t_0}^{t} R(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds, \\
v(t_0) &= u_0 - g(t_1,t_2,\cdots,t_n,u(t_1),u(t_2),\cdots,u(t_n)) \qquad (3.28)
\end{aligned}
$$

is a unique solution to

$$\frac{dv}{dt} = Av(t) + f(t,u(t)) + \int_{t_0}^{t} k(t-s)h(s,u(s))ds, t\in I_{\tilde{T}}. \qquad (3.29)$$

By definition, u is a mild solution to (3.29) on $I_{\tilde{T}}$. By uniqueness of a mild solution to (3.29), we have $u=v$ on $I_{\tilde{T}}$. Thus u satisfies (3.29) and therefore u is a unique classical solution (1.1) on $I_{\tilde{T}}$. Since $\tilde{T},\ t_0<\tilde{T}<T$, arbitrary, u is a classical solution to (1.1) on I_T. This completes the proof.

4 Modified Results

In this section, we study the special case when $\|R(t)\|_{B(E)} \leq Me^{-\alpha t}, 0 \leq t \leq T$ for some constant $\alpha \geq 0$ and when the nonlocal condition (1.2) is given by (1.7). Since we are going to assume a weaker condition and so we hope to get improved conditions in assumption (3.2) in Theorem 3.1. To find those improved conditions we move as follows: We first prove the existence and uniqueness of mild solution $u(.,v)$ of Cauchy problem

$$u'(t) = A[u(t) + \int_o^t F(t-s)u(s)ds] + f(t,u(t)) + \int_0^t k(t-s)h(s,u(s))ds,$$
$$u(0) = v, \quad 0 \leq t \leq T.$$

for any $v \in E$, and then we define an operator along the curve of $u(.,v)$ and show that the operator is a contraction, and finally conclude that operator gives rise to a mild solution of (1.1)-(1.2) by finding its fixed point.

To prove the desired result we need to assume the following (see [16]):

(H6) For some constant $\alpha > 0$, the resolvent operator of (1.1) with $f \equiv 0$ satisfies

$$\|R(t)\|_{B(E)} \leq Me^{-\alpha t}, \quad 0 \leq t \leq T. \tag{4.1}$$

(H7) Nonlocal condition (1.2) is given by (1.7) and

$$\beta \equiv \alpha - M_0 > 0, \quad M\sum_{i=1}^{p} |c_i|e^{-\beta(t_i - t_0)} < 1. \tag{4.2}$$

(M_0 from 3.1, α, M from 4.1.)

Remark 4.1 Note that conditions given in (H7) are better than conditions given in (H5) in some situations.

We need the following inequality to find our results.

Lemma 4.1 [11] *Let $u(t)$ and $b(t)$ be non negative continuous functions for $t \geq \alpha$, and let*

$$u(t) \leq ae^{-\gamma(t-\alpha)} + \int_\alpha^t e^{-\gamma(t-s)}b(s)u(s)ds, t \geq \alpha, \tag{4.3}$$

where $\alpha \geq 0$ and γ are constants. Then

$$u(t) \leq ae^{(-\gamma(t-\alpha)+\int_\alpha^t b(s)ds)}, \quad t \geq \alpha, \tag{4.4}$$

We will use Lemma 4.1 to prove the uniqueness of a mild solution of (1.1)-(1.2). The result is as follows:

Theorem 4.1 *Let assumptions* (H1)–(H4), (H6) *and* (H7) *be satisfied. Then for every $u_0 \in E$* (1.1)–(1.2) *has a unique mild solution.*

Proof Let $u_0 \in E$ be fixed. Then for any $v \in E$, define an operator $X : C([0,T],E) \to C([0,T],E)$ by

$$(Xu)(t) = R(t-t_0)v + \int_{t_0}^{t} R(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds. \qquad (4.5)$$

Then for $u, w \in C([0,T],E), t \in [0,T]$, we have

$$\begin{aligned}\|(Xu)(t)-(Xw)(t)\| &\leq \| \int_{t_0}^{t} R(t-s)[\{f(s,u(s))-f(s,w(s))\} \\ &\quad + \int_{t_0}^{s} k(s-\tau)\{h(\tau,u(\tau))-h(\tau,w(\tau))\}d\tau]ds\| \\ &\leq \int_{t_0}^{t} \|R(t-s)\|[\|f(s,u(s))-f(s,w(s))\| \\ &\quad + \| \int_{t_0}^{s} k(s-\tau)\{h(\tau,u(\tau))-h(\tau,w(\tau))\}d\tau\|]ds\end{aligned}$$

and hence

$$\|(Xu)(t)-(Xw)(t)\|_E \leq (M_0(t-t_0))\|u-w\|_{C(I_T,E)}. \qquad (4.6)$$

with M_0, defined in (3.1). Using (4.5) and repeated application of the inequality (4.6), we have

$$\|(X^n u)(t)-(X^n w)(t)\|_E \leq \frac{[M_0(t-t_0)]^n}{n!}\|u-w\|_{C(I_T,E)}. \qquad (4.7)$$

Therefore, we have

$$\|(X^n u)-(X^n w)\|_{C(I_T,E))} \leq \frac{[M_0 T-t_0)]^n}{n!}\|u-w\|_{C(I_T,E)}. \qquad (4.8)$$

For n large enough $\frac{[M_0(T-t_0)]^n}{n!} < 1$ and by the well known extension of the Banach contraction principle X has a unique fixed point $u(.,v)$. This u satisfies (2.2) and hence it is a unique mild solution to (1.1) on I_T with $u(t_0) = v$.

Next, define an operator $X_1 : E \to E$ by

$$X_1 v = u_0 - \sum_{i=1}^{p} c_i u(t_i), \qquad (4.9)$$

where $u(.) = u(.,v)$ is the unique fixed point of (4.5). Let $u_i(.) = u_i(.,v_i),\ i = 1,2,$ be the unique fixed point of (4.5) with $u_i(t_0) = v_i$. Now, we have:

$$\|X_1 v_1 - X_1 v_2\|_E \leq \sum_{i=1}^{p} |c_i|(\|u_1(t_i)-u_2(t_i)\|_E. \qquad (4.10)$$

Let $w(.) \equiv u_1(.) - u_2(.)$, then we can rewrite (4.5) as

$$\begin{aligned}\|w(t)\|_E &\le \|R(t-t_0)\|_{B(E)}\|v_1 - v_2\|_E \\ &+ \int_{t_0}^{t} \|R(t-s)\|[\|f(s,u_1(s)) - f(s,u_2(s))\| \\ &+\| \int_{t_0}^{s} k(s-\tau)\{h(\tau,u_1(\tau)) - h(\tau,u_2(\tau))\}d\tau\|]ds \\ &\le M\|v_1 - v_2\|_E e^{-\alpha(t-t_0)} + \int_{t_0}^{t} M_0 e^{-\alpha(t-s)}\|u_1(s) - u_2(s)\|_E ds \\ &\le M\|v_1 - v_2\|_E e^{-\alpha(t-t_0)} + \int_{t_0}^{t} M_0 e^{-\alpha(t-s)}\|w(s)\|_E ds, \quad t \in [t_0, T].\end{aligned}$$

Thus by the lemma (4.1),

$$\|w(t)\|_E \le M\|v_1 - v_2\|_E e^{-(\alpha - M_0)(t-t_0)} \tag{4.11}$$

$$= M\|v_1 - v_2\|_E e^{-\beta(t-t_0)}, \quad t \in [t_0, T]. \tag{4.12}$$

By use of (4.12), we can rewrite (4.10) as:

$$\|X_1 v_1 - X_1 v_2\|_E \le (M \sum_{i=1}^{p} |c_i| e^{-\beta(t_i - t_0)} \|v_1 - v_2\|_E. \tag{4.13}$$

By $(H7)$, X_1 is a contraction operator on E and so X_1 has a unique fixed point $v_0 \in E$.

So, for the unique fixed $u(.,v_0)$ of (4.5) with to $u(t_0) = v_0$, we obtain

$$u(t_0, v_0) = v_0 = u_0 - \sum_{i=1}^{p} c_i u(t_i, v_0). \tag{4.14}$$

This implies that

$$u(t, v_0) = R(t-t_0)[u_0 - \sum_{i=1}^{p} c_i u(t_i, v_0)] + \int_{t_0}^{t} R(t-s) f(s,, u(s, v_0))ds, \quad t \in [t_0, T],$$

and hence, $u(., v_0)$ is a mild solution of (1.1-1.2). At last, we show that mild solutions of (1.1)-(1.2) are unique. Since if $u(.)$ is a mild solution of (1.1)-(1.2) with (1.2) given by (1.7), then

$$u(t_0) = u_0 - \sum_{i=1}^{p} c_i u(t_i)$$

and $u(.)$ is also the mild solution of (1.1) with $v = u(t_0)$.

However, X_1 is the contraction map operator and so (4.7) implies that $u(t_0)$ is uniquely determined by X_1. X is also a contraction operator and fixed point of (4.5) is uniquely determined by $v = u(t_0)$. Therefore, it is clear that mild solutions of (1.1) with (1.7) are unique. This completes the proof.

Similar to Theorem 3.1, we have the following result for the classical solution provided that $f, h : I_T \times E \to E$ are continuously differentiable.

Theorem 4.2 *Let the assumptions* (H1)-(H4), (H6) *and* (H7) *be satisfied and let* $u(.)$ *be the unique mild solution of* (1.1) *and* (1.2) *guaranteed by Theorem* 3.1 *with* (1.2) *being given by* (1.7). *Assume further that*

$$u_0 \in D(A), \quad \sum_{i=1}^{p} c_i u(t_i) \in D(A), \quad f, h \in C^1([t_0, T] \times E, E). \tag{4.15}$$

Then $u(.)$ *gives rise to a unique classical solution of* (1.1) *with* (1.7).

5 Application

1. The case in which $k, F, g \equiv 0$ was considered by I. Segal [4]. Different forms of solutions in this particular case have been considered by Pazy [18] and R. H. Martin [3]. The case when $k, F \equiv 0$ was considered by Ludwik Byszewski [5]. The case when $F, g \equiv 0$ has been considered in D. Bahuguna [2]. Also in the case when $F \equiv 0$, existence and uniqueness results for a solution to 1.1, have been analyzed in [1]. Therefore, our results presented here for the problem (1.1) generalizes the results given in [1]–[5] and [18].

2. Consider the following integrodifferential equation termed as classical heat equation for a material with a memory. Let u be the internal energy and

$$f(t, u(t,x)) + \int_{t_0}^{t} k(t-s)h(s, u(s,x))ds$$

be the external heat with

$$\begin{cases} \alpha(t,x) = -u_x(t,x) - \int_{t_0}^{t} b(t-s)u_x(s,x)ds, \quad \text{Heat flux} \\ u_t(t,x) = \frac{\partial}{\partial x}\alpha(t,x) + f(t,u(t,x)) \\ \qquad\qquad + \int_{t_0}^{t} k(t-s)h(s,u(s,x))ds, \quad \text{Balance equation} \\ u(t_0,x) + \sum_{i=1}^{n} u(t_i,x) = u_0(x). \end{cases} \tag{5.1}$$

We can rewrite (5.1) as

$$\begin{aligned} u_t(t,x) &= \frac{\partial^2}{\partial x^2}[u(t,x) + \int_{t_0}^{t} b(t-s)u(s,x)ds] + f(t,u(t,x)) \\ &\quad + \int_{t_0}^{t} k(t-s)h(s,u(s,x))ds, \quad (t,x) \in [t_0,T] \times [0,1], \qquad (5.2) \\ u(t_0,x) &+ \sum_{i=1}^{n} u(t_i,x) = u_0(x). \end{aligned}$$

This is of the type of (1.1) with $A = \frac{\partial^2}{\partial x^2}$ on $H^2[0,1] \cap H_0^1[0,1]$ which generates the strongly continuous semigroup on $L^2[0,1]$ and $b(t)$ is a continuous function. It can be verified that the conditions of Theorem 3.1 are satisfied and thus our analysis ensures existence and uniqueness of a solution to (5.1).

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