Nonlinear Dynamics and Systems Theory, **10** (1) (2010) 39–54

Operational Calculus in Noncooperative Stochastic Games [†]

J. H. Dshalalow [*] and A. Treerattrakoon

Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, Florida 32901-6975, USA.

Received: March 10, 2009; Revised: January 17, 2010

Abstract: We continue investigating an antagonistic game of two players modeled by stochastic processes describing mutual casualties. The game is observed at some random epochs of time. We consider the paths of the game in which one player loses the game. A related functional in our recent work was expressed in terms of the inverse of two-dimensional Laplace-Carson transform. Using operational calculus we manage to find explicitly inverse transforms of the exit time and casualties to both players upon the exit from the game in terms of modified Bessel functions. All are concluded by numerical examples.

Keywords: *noncooperative stochastic games; fluctuation theory; marked point processes; Poisson process; ruin time; exit time; first passage time; Bessel functions.*

Mathematics Subject Classification (2000): 82B41, 60G51, 60G55, 60G57, 91A10, 91A05, 91A60, 60K05.

1 Introduction

We continue our studies initiated in [3] in which we modeled an antagonistic stochastic game by two marked Poisson processes

$$\mathcal{A} := \sum_{j\geq 1} d_j \varepsilon_{r_j} \text{ and } \mathcal{B} := \sum_{k\geq 1} z_k \varepsilon_{w_k} \tag{1.1}$$

on a filtered probability space $(\Omega, \mathcal{F}(\Omega), \mathfrak{F}_t, P)$ specified by

[†] This research is supported by the US Army Grant No. W911NF-07-1-0121 to Florida Institute of Technology.

[*] Corresponding author: eugene@fit.edu

$$Ee^{-u\mathcal{A}(\cdot)} = e^{\lambda_A|\cdot|[h_A(u)-1]},\ h_A(u) = Ee^{-ud_1},\ Re(u) \geq 0, \tag{1.2}$$

$$Ee^{-v\mathcal{B}(\cdot)} = e^{\lambda_B|\cdot|[h_B(v)-1]},\ h_B(v) = Ee^{-vz_1},\ Re(v) \geq 0, \tag{1.3}$$

representing casualties incurred to players A and B. The game starts with hostile actions initiated by one of the players A or B at times r_1 or w_1 (whichever comes first). The players can exchange with several more strikes before the information is first noticed by an observer at time $t_0 \geq \max\{r_1, w_1\}$. The observation of the game continues after t_0 in accordance with a point renewal process $S = \sum_{i\geq 1} \varepsilon_{t_i}$ and it is further extrapolated to the past moment $t_{-1} := \min\{r_1, w_1\}$ thereby forming an extended observation process $\mathcal{T} = \{t_{-1}, t_0, t_1, \ldots\}$. The entire information on the game is available only upon $\mathcal{T}$ and thus the game is reduced to its embedding $\mathcal{A}_\mathcal{T} \otimes \mathcal{B}_\mathcal{T}$. We stop the game when one of the players is ruined, and of all paths of the game we focused on those where player A loses to player B.

Given independent sub-σ-algebras $\mathcal{F}_\mathcal{A}, \mathcal{F}_\mathcal{B}, \mathcal{F}_S \subseteq \mathcal{F}(\Omega)$ we assume that the processes $\mathcal{A}$, $\mathcal{B}$, and S are, respectively, measurable. Let ξ_i and η_i be the corresponding iid increments of damages to A and B and $\Delta_j = t_j - t_{j-1} \in [\Delta]$ (an equivalent class of r.v.'s), $j = 1, 2, \ldots$, with

$$g(u, v, \theta) := Ee^{-u\xi_j - v\eta_j - \theta\Delta_j},\ Re(u) \geq 0,\ Re(v) \geq 0,\ Re(\theta) \geq 0,\ j \geq 1, \tag{1.4}$$

presumably known. The initial observation is defined as $t_0 = \max\{r_1, w_1\} + \Delta_0$, where $\Delta_0 \in [\Delta]$ and Δ_0 is independent from the rest of the Δ's. The random *exit indices* are

$$\mu := \inf\{j \geq 0 : \alpha_j = \alpha_0 + \xi_1 + \ldots + \xi_j > M\}, \tag{1.5}$$

$$\nu := \inf\{k \geq 0 : \beta_k = \beta_0 + \eta_1 + \ldots + \eta_k > N\}, \tag{1.6}$$

with α_0 and β_0 being the casualties to A and B at t_0, and M and N are respective tolerance thresholds. Related on μ and ν are the following r.v.'s:

t_μ is the nearest observation epoch when player A's damages exceed threshold M.

t_ν is the first observation of $\mathcal{T}$ when player B's damages exceed threshold N.

Apparently, α_μ and β_ν are the respective cumulative damages to players A and B at their ruin times. We will be concerned, however, with the ruin time of player A and thus restrict our game to the trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$. Accordingly, we studied in [3], among other things,

$$\varphi_\mu = \varphi_\mu(u, v, \vartheta) = E[e^{-u\alpha_\mu - v\beta_\mu - \theta t_\mu} \mathbf{1}_{\{\mu<\nu\}}] \tag{1.7}$$

and obtained

$$\varphi_\mu = \mathcal{LC}_{xy}^{-1}\Big(\phi_0(x, 0, 0, u, v+y, \theta) - \phi_0(0, 0, 0, u+x, v+y, \theta)\frac{1 - g(u, v+y, \theta)}{1 - g(u+x, v+y, \theta)}\Big), \tag{1.8}$$

where $\mathcal{LC}^{-1}$ is the inverse of the two-dimensional Laplace–Carson transform

$$\mathcal{LC}_{pq}(\cdot)(x, y) := xy \int_{p=0}^{\infty} \int_{q=0}^{\infty} e^{-xp-yq}(\cdot) d(p, q),\ Re(x) > 0,\ Re(y) > 0, \tag{1.9}$$

According to [3],

$$\begin{aligned}\phi_0(x,0,0,u,v,\theta) &= E[e^{-x\alpha_{-1}-u\alpha_0-v\beta_0-\theta t_0}]\\ &= \frac{\lambda_A\lambda_B\delta(\theta^*)}{\theta+\lambda_A+\lambda_B}\Big(\frac{1}{\theta_A+\lambda_B}h_A(x+u)h_B(v)+\frac{1}{\theta_B+\lambda_A}h_A(u)h_B(v)\Big),\end{aligned} \tag{1.10}$$

$$\begin{aligned}\phi_0(0,0,0,u,v,\theta) &= E[e^{-u\alpha_0-v\beta_0-\theta t_0}]\\ &= \frac{\lambda_A\lambda_B\delta(\theta^*)}{\theta+\lambda_A+\lambda_B}\Big(\frac{1}{\theta_A+\lambda_B}h_A(u)h_B(v)+\frac{1}{\theta_B+\lambda_A}h_A(u)h_B(v)\Big),\end{aligned} \tag{1.11}$$

and

$$\theta_0^* := \theta+\lambda_A(1-h_A(u))+\lambda_B(1-h_B(v)), \delta(\theta) := Ee^{-\theta\Delta}, \tag{1.12}$$

$$\theta_A := \theta-\lambda_A(h_A(u)-1), \theta_B := \theta-\lambda_B(h_B(v)-1). \tag{1.13}$$

The involvement of the inverse of the Laplace–Carson transform in (1.8) at first does not look like the above formulas are analytically tractable. We demonstrate that this is not the case and consider a number of special cases (of independent interest) which are all Laplace–Carson invertible and thus provide the first vivid argument for analytical tractability of the results obtained in [3]. They are shown to be numerically tame and as such are rendered by trivial computational procedures. Most of them are reduced to definite integrals of the modified Bessel functions. In one case an explicit marginal probability density function is obtained. The original MATLAB routine is also attached.

2 A Special Case

We assume that the intervals $\Delta_0, \Delta_1, \ldots$ between the successive observation times $t_0, t_1, \ldots$, are exponentially distributed with parameter δ, i.e.

$$\delta(\theta) := Ee^{-\theta\Delta_0} = \frac{\delta}{\delta+\theta}. \tag{2.1}$$

Furthermore, we assume that the marks in the processes $\mathcal{A}$ and $\mathcal{B}$ specified by h_A and h_B in (1.2) and (1.3), respectively, are exponential with parameters h and H, i.e.

$$h_A(u) = \frac{h}{h+u} \text{ and } h_B(v) = \frac{H}{H+v}. \tag{2.2}$$

(1.8) for this special case reduces to a form for which we can find the Laplace–Carson inverse explicitly. We start with the first factor, $\phi_0(x,0,0,u,v+y,\theta)$ of (1.10):

$$\begin{aligned}&\phi_0(x,0,0,u,v+y,\theta)\\ &= \frac{\lambda_A\lambda_B h_B(v+y)}{\theta+\lambda_A+\lambda_B}\times\delta(\theta+\lambda_A(1-h_A(u))+\lambda_B(1-h_B(v+y)))\\ &\quad\times\Big(\frac{1}{\theta+\lambda_B-\lambda_A(h_A(u)-1)}h_A(u+x)+\frac{1}{\theta+\lambda_A-\lambda_B(h_B(v+y)-1)}h_A(u)\Big).\end{aligned} \tag{2.3}$$

Continuing with calculations, after some algebra, we arrive at

$$\begin{aligned}
&\phi_0(x,0,0,u,v+y,\theta)\\
&=\frac{\lambda_A\lambda_B hH}{\theta+\lambda_A+\lambda_B}\\
&\quad\times\frac{\delta(h+u)}{(H+v+y)[(\delta+\theta)(h+u)+\lambda_A u+\lambda_B(h+u)]-\lambda_B H(h+u)}\\
&\quad\times\Big(\frac{1}{(\theta+\lambda_B)(h+u)+\lambda_A u}+\frac{-x}{(\theta+\lambda_B)(h+u)+\lambda_A u}\cdot\frac{1}{h+u+x}\\
&\qquad+\frac{H+v+y}{(\theta+\lambda_A+\lambda_B)(H+v+y)-\lambda_B H}\cdot\frac{1}{h+u}\Big).
\end{aligned}\tag{2.4}$$

Now we apply the Laplace–Carson inverse to (2.4):

$$\mathcal{LC}_{xy}^{-1}(\phi_0(x,0,0,u,v+y,\theta))(p,q)$$

or proceed with $\mathfrak{L}_{xy}^{-1}$ being the two-dimensional Laplace inverse, in the form

$$=\mathfrak{L}_{xy}^{-1}(\tfrac{1}{xy}\cdot\phi_0(x,0,0,u,v+y,\theta))(p,q)$$

(by Fubini's Theorem, we can apply single-variate Laplace inverses first in x and later on in y)

$$\begin{aligned}
&=\mathfrak{L}_y^{-1}\Big\{\frac{\lambda_A\lambda_B hH\delta(h+u)}{\widetilde{\theta}G_1}\cdot\frac{1}{y}\cdot\frac{1}{H+v+y-\frac{\lambda_B H(h+u)}{G_1}}\\
&\quad\times\Big(\frac{1}{G_2}+\frac{-1}{G_2}\cdot e^{-p(h+u)}+\frac{1}{\widetilde{\theta}(h+u)}+\frac{\lambda_B H}{\widetilde{\theta}^2(h+u)}\cdot\frac{1}{H+v+y-\frac{\lambda_B H}{\widetilde{\theta}}}\Big)\Big\}(q),
\end{aligned}$$

then

$$\begin{aligned}
&=\frac{\lambda_A\lambda_B hH\delta}{\widetilde{\theta}}\cdot\frac{1}{(H+v)G_1-\lambda_B H(h+u)}\\
&\quad\times\Big(\frac{h+u}{G_2}+\frac{H+v}{\widetilde{\theta}(H+v)-\lambda_B H}-\frac{h+u}{G_2}\cdot e^{-p(h+u)}\\
&\qquad+\Big\{\frac{-\lambda_B H}{\widetilde{\theta}}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}+\frac{-G_1}{\widetilde{\theta}G_3}\Big\}e^{-q(H+v-\frac{\lambda_B H}{\widetilde{\theta}})}\\
&\qquad+\Big\{\frac{-(h+u)}{G_2}+\frac{h+u}{G_3}+\frac{h+u}{G_2}\cdot e^{-p(h+u)}\Big\}e^{-q(H+v-\frac{\lambda_B H(h+u)}{G_1})}\Big),
\end{aligned}\tag{2.5}$$

where

$$\widetilde{\theta}=\theta+\lambda_A+\lambda_B,\tag{2.6}$$

$$G_1=(\delta+\widetilde{\theta})(h+u)-\lambda_A h,\ G_2=\widetilde{\theta}(h+u)-\lambda_A h,\tag{2.7}$$

$$G_3=\delta(h+u)-\lambda_A h.\tag{2.8}$$

We turn to the second term $\phi_0(0,0,0,u+x,v+y,\theta)\frac{1-g(u,v+y,\theta)}{1-g(u+x,v+y,\theta)}$ of (1.8). Continuing with similar but more tedious calculations, we have its Laplace-Carson inverse in variable x :

$$\mathcal{LC}_x^{-1}\Big\{\phi_0(0,0,0,u+x,v+y,\theta)\frac{1-g(u,v+y,\theta)}{1-g(u+x,v+y,\theta)}\Big\}(p)$$
$$= \Lambda\big(1-g(u,v+y,\theta)\big)\cdot y$$
$$\times\Bigg[\Big(\frac{-1}{\lambda_A h}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}+\{\frac{1}{\widetilde{\theta}}+\frac{G_2}{\lambda_A h\widetilde{\theta}}+\frac{h+u}{G_2}\}\frac{1}{G_2(H+v)-\lambda_B H(h+u)}\Big)\frac{1}{y}$$
$$+\frac{1}{\lambda_A h}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}\cdot\frac{1}{H+v+y-\frac{\lambda_B H}{\widetilde{\theta}}}$$
$$-\{\frac{1}{\widetilde{\theta}}+\frac{G_2}{\lambda_A h\widetilde{\theta}}+\frac{h+u}{G_2}\}\frac{1}{G_2(H+v)-\lambda_B H(h+u)}\cdot\frac{1}{H+v+y-\frac{\lambda_B H(h+u)}{G_2}}$$
$$+\frac{1}{\lambda_B H G_2}\cdot\frac{1}{y}\cdot e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})}$$
$$+\Bigg[\Big(\frac{-1}{\lambda_B H G_2}+\frac{1}{\lambda_A h}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}$$
$$+\{\frac{-1}{\widetilde{\theta}}+\frac{-G_2}{\lambda_A h\widetilde{\theta}}+\frac{-(h+u)}{G_2}\}\frac{1}{G_2(H+v)-\lambda_B H(h+u)}\Big)\frac{1}{y}$$
$$+\frac{-1}{\lambda_A h}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}\cdot\frac{1}{H+v+y-\frac{\lambda_B H}{\widetilde{\theta}}}$$
$$+\Big\{\frac{1}{\widetilde{\theta}}+\frac{G_2}{\lambda_A h\widetilde{\theta}}+\frac{h+u}{G_2}\Big\}\frac{1}{G_2(H+v)-\lambda_B H(h+u)}$$
$$\times\frac{1}{H+v+y-\frac{\lambda_B H(h+u)}{G_2}}\Bigg]e^{-p(h+u+\frac{D(y)}{C'(y)})}\Bigg], \tag{2.9}$$

where

$$\Lambda=\frac{\lambda_A\lambda_B hH\delta}{\widetilde{\theta}},\ C'(y)=\widetilde{\theta}(H+v+y)-\lambda_B H,\ D(y)=-\lambda_A h(H+v+y). \tag{2.10}$$

Now, we will apply the single-variate Laplace-Carson inverse in y to (2.9). We will use the following formula for the Laplace inverse (cf. [1, 2]):

$$\mathfrak{L}_y^{-1}(\frac{1}{y+b_2}\cdot e^{\frac{a}{y+b_1}})(q)=e^{-b_1 q}\mathcal{I}_0(2\sqrt{aq})$$
$$+(b_1-b_2)\cdot e^{-b_2 q}\int_{z=0}^{q}e^{(b_2-b_1)z}\mathcal{I}_0(2\sqrt{az})dz, \tag{2.11}$$

where $\mathcal{I}_0$ is the modified Bessel function of order zero. Using (2.11) in (2.9) in combination with (2.5), we finally have

$$\varphi_\mu(u,v,\theta):=E\Big[e^{-u\alpha_\mu-v\beta_\mu-\theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big]$$
$$=\mathcal{L}_{xy}^{-1}(\phi_0(x,0,0,u,v+y,\theta))(p,q)$$
$$-\mathcal{L}_{xy}^{-1}\Big(\phi_0(0,0,0,u+x,v+y,\theta)\frac{1-g(u,v+y,\theta)}{1-g(u+x,v+y,\theta)}\Big)(p,q)$$

$$
\begin{aligned}
&= \frac{-\lambda_A\lambda_B hH\delta}{\widetilde{\theta}G_2}\cdot\frac{h+u}{(H+v)G_1-\lambda_B H(h+u)}\cdot e^{-p(h+u)}(1-e^{-q(H+v-\frac{\lambda_B H(h+u)}{G_1})})\\
&\quad+\Big(\frac{-\lambda_A h\delta}{\widetilde{\theta}}\cdot\frac{H+v}{(H+v)G_1-\lambda_B H(h+u)}+\frac{\lambda_A\lambda_B hH\delta}{\widetilde{\theta}G_2}\cdot\frac{h+u}{(H+v)G_1-\lambda_B H(h+u)}\Big)\\
&\quad\times e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})}\\
&\quad+\Big(\frac{-\lambda_A h\delta}{\widetilde{\theta}G_1}+\frac{\lambda_A h\delta}{\widetilde{\theta}}\cdot\frac{H+v}{(H+v)G_1-\lambda_B H(h+u)}\\
&\qquad-\frac{\lambda_A\lambda_B hH\delta}{\widetilde{\theta}G_2}\cdot\frac{h+u}{(H+v)G_1-\lambda_B H(h+u)}\Big)\cdot e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})}\cdot e^{-q(H+v-\frac{\lambda_B H(h+u)}{G_1})}\\
&\quad+\frac{\lambda_A h\delta}{\widetilde{\theta}G_1}\cdot e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})}\cdot e^{-q(H+v-\frac{\lambda_B H}{\widetilde{\theta}})}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{\widetilde{\theta}^2}}\Big)\\
&\quad+\frac{\lambda_A h\delta}{\widetilde{\theta}}\cdot\frac{(H+v)^2}{(H+v)G_1-\lambda_B H(h+u)}\cdot e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})}\\
&\quad\times\int_{z=0}^{q} e^{-(H+v-\frac{\lambda_B H}{\widetilde{\theta}})z}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{\widetilde{\theta}^2}}\Big)dz\\
&\quad+\Big(\frac{\lambda_A\lambda_B hH\delta(h+u)}{\widetilde{\theta}G_1^2}+\frac{-\lambda_A\lambda_B hH\delta(h+u)}{\widetilde{\theta}G_1}\cdot\frac{H+v}{(H+v)G_1-\lambda_B H(h+u)}\Big)\\
&\quad\times e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})}\cdot e^{-q(H+v-\frac{\lambda_B H(h+u)}{G_1})}\\
&\quad\times\int_{z=0}^{q} e^{(\frac{\lambda_B HG_3}{\widetilde{\theta}G_1})z}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{\widetilde{\theta}^2}}\Big)dz, \qquad (2.12)
\end{aligned}
$$

where G_1, G_2, G_3 are defined in (2.7-2.8).

3 Marginal Functionals

Our next goal is to get marginal transforms. This can be directly obtained from $\varphi_\mu(u,v,\theta)$ of (2.12).

Special case 1, with $v=\theta=0$ we have the marginal Laplace-Stieltjes transform of the amount of casualties to player A (who is supposed to lose) at the exit of the game:

$$\varphi_\mu(u,0,0) := E\Big[e^{-u\alpha_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big]. \qquad (3.1)$$

Correspondingly, we modify the above components in (2.12) to

$$\widetilde{\theta}=\lambda_A+\lambda_B, \qquad (3.2)$$

$$G_1=(\delta+\lambda_A+\lambda_B)(h+u-\frac{\lambda_A h}{\delta+\lambda_A+\lambda_B}), \qquad (3.3)$$

$$G_2=(\lambda_A+\lambda_B)(h+u-\frac{\lambda_A h}{\lambda_A+\lambda_B}), \qquad (3.4)$$

$$h+u-\frac{\lambda_A h}{\widetilde{\theta}}=u+\frac{\lambda_B h}{\lambda_A+\lambda_B}, \qquad (3.5)$$

$$H + v - \frac{\lambda_B H(h+u)}{G_1} = \frac{H(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B} + \frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}, \tag{3.6}$$

$$H + v + \frac{\lambda_B H}{\widetilde{\theta}} = \frac{\lambda_A H}{\lambda_A+\lambda_B}, \tag{3.7}$$

$$\begin{aligned}\frac{\lambda_B H G_3}{\widetilde{\theta} G_1} &= \frac{\lambda_B H\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)} \\ &\quad + \frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}},\end{aligned} \tag{3.8}$$

$$\frac{1}{(H+v)G_1 - \lambda_B H(h+u)} = \frac{1}{H(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}, \tag{3.9}$$

$$\widetilde{\theta}(H+v) - \lambda_B H = \lambda_A H. \tag{3.10}$$

Substituting (3.2-3.10) into (2.12), we arrive at

$$\begin{aligned}
\varphi_\mu(u,0,0) &= \Big(\frac{-\lambda_A\lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\lambda_A+\lambda_B}}\cdot\frac{h+u}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\cdot e^{-p(h+u)}\Big)\\
&\quad\times\Big[1-\exp\Big(-q\big(\frac{H(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}\big)\Big)\Big]\\
&\quad+\Big(\frac{-\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\\
&\qquad+\frac{\lambda_A\lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\lambda_A+\lambda_B}}\cdot\frac{h+u}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\Big)\cdot e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\\
&\quad+\Big(\frac{-\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A+\lambda_B}}\\
&\qquad+\frac{\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\\
&\qquad+\frac{-\lambda_A\lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\lambda_A+\lambda_B}}\cdot\frac{h+u}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\Big)\cdot e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\\
&\quad\times\exp\Big(-q\big(\frac{H(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}\big)\Big)\\
&\quad+\frac{\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A+\lambda_B}}\\
&\quad\times e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}})\\
&\quad+\frac{\lambda_A hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\cdot e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\\
&\quad\times\int_{z=0}^{q} e^{-(\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz
\end{aligned}$$

$$+\Big(\frac{\lambda_A\lambda_B hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{h+u}{(h+u-\frac{\lambda_A h}{\delta+\lambda_A+\lambda_B})^2}$$
$$+\frac{-\lambda_A\lambda_B hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A+\lambda_B}}\cdot\frac{h+u}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\Big)$$
$$\times e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\exp\Big(-q\big(\frac{H(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}\big)\Big)$$
$$\times\int_{z=0}^{q}\exp\Big[\big(\frac{\lambda_B H\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}\big)z\Big]$$
$$\times\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\Big)dz. \tag{3.11}$$

Special case 2. Setting $u=\theta=0$ gets us to the marginal Laplace-Stiltjes transform of the casualties to player B at the exit of the game to be lost by player A:

$$\varphi_\mu(0,v,0):=E\Big[e^{-v\beta_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big]. \tag{3.12}$$

The Laplace inverse formula (cf. [1, 2]) that we will use along with (2.11) is:

$$\mathcal{L}_y^{-1}\Big(\frac{e^{\frac{a}{y+b}}}{(y+b)^2}\Big)(q)=\sqrt{\frac{q}{a}}\cdot e^{-bq}\mathcal{I}_1(2\sqrt{aq}), \tag{3.13}$$

where $\mathcal{I}_1$ is the modified Bessel function of order one. After setting $u=\theta=0$ in (1.8), we arrive at

***(i)* Case $\delta\neq\lambda_A$,**

$$\varphi_\mu^1(0,v,0)=\frac{-\lambda_A H\delta}{\lambda_A+\lambda_B}\cdot\frac{1}{H\delta+(\delta+\lambda_B)v}\cdot e^{-ph}$$
$$+\frac{\lambda_A H\delta}{\lambda_A+\lambda_B}\cdot\frac{1}{H\delta+(\delta+\lambda_B)v}\cdot e^{-ph}\cdot e^{-q(v+\frac{H\delta}{\delta+\lambda_B})}$$
$$+\Big(\frac{-\lambda_A\delta}{\lambda_A+\lambda_B}\cdot\frac{v}{H\delta+(\delta+\lambda_B)v}$$
$$+\frac{-\lambda_A H\delta^2}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot\frac{1}{H\delta+(\delta+\lambda_B)v}\cdot e^{-q(v+\frac{H\delta}{\delta+\lambda_B})}\Big)\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}$$
$$+\frac{\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}\Big)$$
$$+\frac{\lambda_A\delta}{\lambda_A+\lambda_B}\cdot\frac{(H+v)^2}{H\delta+(\delta+\lambda_B)v}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\int_{z=0}^{q}e^{-(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\Big)dz$$
$$+\frac{-\lambda_A\lambda_B^2H^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^2}\cdot\frac{1}{H\delta+(\delta+\lambda_B)v}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(v+\frac{H\delta}{\delta+\lambda_B})}$$
$$\times\int_{z=0}^{q}e^{(\frac{\lambda_B H(\delta-\lambda_A)}{(\lambda_A+\lambda_B)(\delta+\lambda_B)})z}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\Big)dz. \tag{3.14}$$

***(ii)* Case $\delta = \lambda_A$,**

$$\varphi_\mu^2(0,v,0) = \Big(\frac{-\lambda_A^2 H}{\lambda_A+\lambda_B}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v} + \frac{\lambda_A^2 H}{\lambda_A+\lambda_B}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})}\Big)\cdot e^{-ph}$$
$$+\Big(\frac{-\lambda_A^2 v}{\lambda_A+\lambda_B}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v} + \frac{-\lambda_A^3 H}{(\lambda_A+\lambda_B)^2}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})}\Big)\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}$$
$$+\frac{\lambda_A^2}{(\lambda_A+\lambda_B)^2}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}})$$
$$+\frac{\lambda_A^2}{\lambda_A+\lambda_B}\cdot\frac{(H+v)^2}{\lambda_A H+(\lambda_A+\lambda_B)v}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\times\int_{z=0}^{q} e^{-(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz$$
$$+\frac{-\lambda_A^2\lambda_B^2 H^2}{(\lambda_A+\lambda_B)^2}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v}\sqrt{\frac{q}{\lambda_A\lambda_B hHp}}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})}\times\mathcal{I}_1(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}). \tag{3.15}$$

Special case 3, with $u = v = 0$ look into the Laplace-Stieltjes transform of the exit time of the game to be lost by player A:

$$\varphi_\mu(0,0,\theta) := E\Big[e^{-\theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big]. \tag{3.16}$$

Proceeding similarly as special case 1, we have

$$\varphi_\mu(0,0,\theta) = \frac{-\lambda_A\lambda_B\delta}{\widetilde{\theta}(\theta+\lambda_B)(\delta+\theta)}\cdot e^{-ph} + \frac{\lambda_A\lambda_B\delta}{\widetilde{\theta}(\theta+\lambda_B)(\delta+\theta)}\cdot e^{-ph}\cdot e^{-q(\frac{(\delta+\theta)H}{\delta+\theta+\lambda_B})}$$
$$+\frac{-\lambda_A\delta\theta}{\widetilde{\theta}(\theta+\lambda_B)(\delta+\theta)}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})}$$
$$+\frac{-\lambda_A\lambda_B\delta^2}{\widetilde{\theta}(\theta+\lambda_B)(\delta+\theta)(\delta+\theta+\lambda_B)}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})}\cdot e^{-q(\frac{(\delta+\theta)H}{\delta+\theta+\lambda_B})}$$
$$+\frac{\lambda_A\delta}{\widetilde{\theta}(\delta+\theta+\lambda_B)}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})}\cdot e^{-q(\frac{(\theta+\lambda_A)H}{\widetilde{\theta}})}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{\widetilde{\theta}^2}})$$
$$+\frac{\lambda_A H\delta}{\widetilde{\theta}(\delta+\theta)}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})}\int_{z=0}^{q} e^{-(\frac{(\theta+\lambda_A)H}{\widetilde{\theta}})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{\widetilde{\theta}^2}})dz$$
$$+\frac{-\lambda_A\lambda_B^2 H\delta}{\widetilde{\theta}(\delta+\theta)(\delta+\theta+\lambda_B)^2}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})}\cdot e^{-q(\frac{(\delta+\theta)H}{\delta+\theta+\lambda_B})}\times\int_{z=0}^{q} e^{(\frac{\lambda_B H(\delta-\lambda_A)}{\widetilde{\theta}(\delta+\theta+\lambda_B)})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{\widetilde{\theta}^2}})dz. \tag{3.17}$$

4 The Explicit Distribution of the Casualties Value to Player A

Now, we can find the pdf of the exit value of casualties to player A (special case 1) by taking the inverse Laplace transform w.r.t. variable u. We distinguish two cases which are $\delta \neq \lambda_B$ and $\delta = \lambda_B$, respectively. The Laplace inverse formulas that we will use along with (2.11) are:

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{1}{y+b})(q) = e^{-b(q-\alpha)} \mathbf{1}_{(\alpha,\infty)}(q), \tag{4.1}$$

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{1}{(y+b)^2})(q) = (q-\alpha) e^{-b(q-\alpha)} \mathbf{1}_{(\alpha,\infty)}(q), \tag{4.2}$$

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b}}}{y+b})(q) = e^{-b(q-\alpha)} \mathcal{I}_0(2\sqrt{a(q-\alpha)}) \mathbf{1}_{(\alpha,\infty)}(q), \tag{4.3}$$

$$\begin{aligned} \mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b_1}}}{y+b_2})(q) &= e^{-b_1(q-\alpha)} \mathcal{I}_0(2\sqrt{a(q-\alpha)}) \mathbf{1}_{(\alpha,\infty)}(q) \\ &+ (b_1 - b_2) \cdot e^{-b_2(q-\alpha)} \int_{z=0}^{q-\alpha} e^{(b_2-b_1)z} \mathcal{I}_0(2\sqrt{az}) dz \mathbf{1}_{(\alpha,\infty)}(q), \end{aligned} \tag{4.4}$$

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b}}}{(y+b)^2})(q) = \sqrt{\frac{q-\alpha}{a}} \cdot e^{-b(q-\alpha)} \mathcal{I}_1(2\sqrt{a(q-\alpha)}) \mathbf{1}_{(\alpha,\infty)}(q), \tag{4.5}$$

$$\begin{aligned} \mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b_1}}}{(y+b_2)^2})(q) &= e^{-b_2(q-\alpha)} \int_{z=0}^{q-\alpha} e^{(b_2-b_1)z} \mathcal{I}_0(2\sqrt{az}) dz \mathbf{1}_{(\alpha,\infty)}(q) \\ &+ (b_1 - b_2) \cdot e^{-b_2(q-\alpha)} \int_{z=0}^{q-\alpha} (q-\alpha-z) \cdot e^{(b_2-b_1)z} \mathcal{I}_0(2\sqrt{az}) dz \mathbf{1}_{(\alpha,\infty)}(q). \end{aligned} \tag{4.6}$$

Equations (4.4) and (4.6) can be readily proved, while the rest of the above formulas can be found in references [1, 2].
After that, we apply the Laplace inverse in (3.11), arriving at

***(i)* Case $\boldsymbol{\delta \neq \lambda_B}$,**

$$\begin{aligned} \mathcal{L}_u^{-1}\Big\{\varphi_\mu^1(u,0,0)\Big\}(s) &= \frac{\lambda_A \lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta-\lambda_B)} \cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}} \Big(1 - e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}\Big) \mathbf{1}_{(p,\infty)}(s) \\ &+ \Big(\frac{\lambda_A \lambda_B h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta-\lambda_B)} \cdot e^{-ph} + \frac{-\lambda_A h\delta^2}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta-\lambda_B)} \cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\Big) \\ &\times e^{-\frac{h\delta(s-p)}{\delta+\lambda_A}} \mathbf{1}_{(p,\infty)}(s) \\ &+ \Big(\frac{\lambda_A \lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)} \cdot e^{-ph} + \frac{-\lambda_A \lambda_B h\delta^2}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)} \cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\Big) \\ &\times e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}} \cdot e^{-\frac{h(\delta+\lambda_B)(s-p)}{\delta+\lambda_A+\lambda_B}} \mathcal{I}_0(2\sqrt{\frac{\lambda_A \lambda_B hHq(s-p)}{(\delta+\lambda_A+\lambda_B)^2}}) \mathbf{1}_{(p,\infty)}(s) \\ &+ \frac{\lambda_A^2 \lambda_B h^2 \delta^2}{(\lambda_A+\lambda_B)^3(\delta-\lambda_B)(\delta+\lambda_A+\lambda_B)} \cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}} \cdot e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}} \Big(e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}} - 1\Big) \\ &\times \int_{w=0}^{s-p} e^{(\frac{\lambda_B h}{\lambda_A+\lambda_B} - \frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B})w} \mathcal{I}_0(2\sqrt{\frac{\lambda_A \lambda_B hHqw}{(\delta+\lambda_A+\lambda_B)^2}}) dw \mathbf{1}_{(p,\infty)}(s) \end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{-\lambda_A^2\lambda_B^2h^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)^2(\delta-\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot e^{-ph} \\
&\quad+\frac{\lambda_A^2\lambda_B h^2\delta^2}{(\lambda_A+\lambda_B)(\delta+\lambda_A)^2(\delta-\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\Big)\cdot e^{-\frac{h\delta(s-p)}{\delta+\lambda_A}} \\
&\quad\times e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}}\int_{w=0}^{s-p}e^{(\frac{h\delta}{\delta+\lambda_A}-\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B})w}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHqw}{(\delta+\lambda_A+\lambda_B)^2}})dw\mathbf{1}_{(p,\infty)}(s) \\
&+\frac{\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{\lambda_A Hq}{\lambda_A+\lambda_B}}\cdot e^{-\frac{h(\delta+\lambda_B)(s-p)}{\delta+\lambda_A+\lambda_B}} \\
&\quad\times\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}})\mathbf{1}_{(p,\infty)}(s) \\
&+\frac{\lambda_A hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{h\delta(s-p)}{\delta+\lambda_A}} \\
&\quad\times\int_{z=0}^{q}e^{-\frac{\lambda_A Hz}{\lambda_A+\lambda_B}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz\mathbf{1}_{(p,\infty)}(s) \\
&+\int_{z=0}^{q}\Big[\Big(\frac{-\lambda_A\lambda_B^2hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)^2}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)(s-p)}{(\delta+\lambda_A+\lambda_B)^2}}) \\
&\qquad+\frac{\lambda_A^2\lambda_B h^2H\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^2}\sqrt{\frac{s-p}{\lambda_A\lambda_B hH(q-z)}} \\
&\qquad\times\mathcal{I}_1(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)(s-p)}{(\delta+\lambda_A+\lambda_B)^2}})\Big)\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}} \\
&\quad\times e^{-\frac{h(\delta+\lambda_B)(s-p)}{\delta+\lambda_A+\lambda_B}}\cdot e^{\frac{\lambda_B H\delta z}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})\mathbf{1}_{(p,\infty)}(s) \\
&\quad+\frac{-\lambda_A^2\lambda_B h^2H\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)^2(\delta+\lambda_A+\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{h\delta(s-p)}{\delta+\lambda_A}}\cdot e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}} \\
&\qquad\times e^{\frac{\lambda_B H\delta z}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\int_{w=0}^{s-p}e^{(\frac{h\delta}{\delta+\lambda_A}-\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B})w} \\
&\qquad\times\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)w}{(\delta+\lambda_A+\lambda_B)^2}})\,dw\mathbf{1}_{(p,\infty)}(s)\Big]dz. \qquad (4.7)
\end{aligned}
$$

***(ii)* Case $\boldsymbol{\delta=\lambda_B}$,**

$$
\begin{aligned}
&\mathcal{L}_u^{-1}\Big\{\varphi_\mu^2(u,0,0)\Big\}(s)=\Big(\frac{-\lambda_A^2\lambda_B h}{(\lambda_A+\lambda_B)^3}+\frac{\lambda_A^2\lambda_B^2h^2(s-p)}{(\lambda_A+\lambda_B)^4}+\frac{-\lambda_A\lambda_B^2h}{(\lambda_A+\lambda_B)^3}\cdot e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}} \\
&\quad+\frac{-\lambda_A^2\lambda_B^2h^2(s-p)}{(\lambda_A+\lambda_B)^4}\cdot e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}\Big)\cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}}\mathbf{1}_{(p,\infty)}(s) \\
&+\frac{\lambda_A\lambda_B h}{(\lambda_A+\lambda_B)(\lambda_A+2\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{\lambda_A Hq}{\lambda_A+\lambda_B}}\cdot e^{-\frac{2\lambda_B h(s-p)}{\lambda_A+2\lambda_B}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}})\mathbf{1}_{(p,\infty)}(s)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{\lambda_A\lambda_B^2 h}{(\lambda_A+\lambda_B)^3}\cdot e^{-ph}+\frac{-\lambda_A\lambda_B^3 h}{(\lambda_A+\lambda_B)^3(\lambda_A+2\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\Big)\\
&\quad\times e^{-\frac{Hq(\lambda_A+\lambda_B)}{\lambda_A+2\lambda_B}}\cdot e^{-\frac{2\lambda_B h(s-p)}{\lambda_A+2\lambda_B}}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHq(s-p)}{(\lambda_A+2\lambda_B)^2}}\Big)\mathbf{1}_{(p,\infty)}(s)\\
&+\frac{\lambda_A\lambda_B hH}{(\lambda_A+\lambda_B)^2}\cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}}\int_{z=0}^{q}e^{-\frac{\lambda_A Hz}{\lambda_A+\lambda_B}}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\Big)dz\mathbf{1}_{(p,\infty)}(s)\\
&+\Big(\frac{-2\lambda_A^2\lambda_B^3h^2}{(\lambda_A+\lambda_B)^4(\lambda_A+2\lambda_B)}+\frac{\lambda_A^2\lambda_B^2h^2(\lambda_A+3\lambda_B)}{(\lambda_A+\lambda_B)^4(\lambda_A+2\lambda_B)}\cdot e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}\Big)\cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}}\\
&\quad\times e^{-\frac{Hq(\lambda_A+\lambda_B)}{\lambda_A+2\lambda_B}}\int_{w=0}^{s-p}e^{(\frac{\lambda_B h}{\lambda_A+\lambda_B}-\frac{2\lambda_B h}{\lambda_A+2\lambda_B})w}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHqw}{(\lambda_A+2\lambda_B)^2}}\Big)dw\mathbf{1}_{(p,\infty)}(s)\\
&+\Big(\frac{-\lambda_A^3\lambda_B^3h^3}{(\lambda_A+\lambda_B)^5(\lambda_A+2\lambda_B)}+\frac{\lambda_A^3\lambda_B^3h^3}{(\lambda_A+\lambda_B)^5(\lambda_A+2\lambda_B)}e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}\Big)\\
&\quad\times e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}}\cdot e^{-\frac{Hq(\lambda_A+\lambda_B)}{\lambda_A+2\lambda_B}}\\
&\quad\times\int_{w=0}^{s-p}(s-p-w)\cdot e^{(\frac{\lambda_B h}{\lambda_A+\lambda_B}-\frac{2\lambda_B h}{\lambda_A+2\lambda_B})w}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHqw}{(\lambda_A+2\lambda_B)^2}}\Big)dw\mathbf{1}_{(p,\infty)}(s)\\
&+\int_{z=0}^{q}\Big[\Big(\frac{-\lambda_A\lambda_B^3 hH}{(\lambda_A+\lambda_B)^2(\lambda_A+2\lambda_B)^2}\cdot e^{-\frac{2\lambda_B h(s-p)}{\lambda_A+2\lambda_B}}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)(s-p)}{(\lambda_A+2\lambda_B)^2}}\Big)\\
&\qquad+\frac{\lambda_A^2\lambda_B^2h^2H}{(\lambda_A+\lambda_B)(\lambda_A+2\lambda_B)^2}\sqrt{\frac{s-p}{\lambda_A\lambda_B hH(q-z)}}\cdot e^{-\frac{2\lambda_B h(s-p)}{\lambda_A+2\lambda_B}}\\
&\qquad\times\mathcal{I}_1\Big(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)(s-p)}{(\lambda_A+2\lambda_B)^2}}\Big)\\
&\qquad+\frac{-\lambda_A^2\lambda_B^2h^2H}{(\lambda_A+\lambda_B)^3(\lambda_A+2\lambda_B)}\cdot e^{-\frac{\lambda_B h(s-p)}{\lambda_A+\lambda_B}}\\
&\qquad\times\int_{w=0}^{s-p}e^{(\frac{\lambda_B h}{\lambda_A+\lambda_B}-\frac{2\lambda_B h}{\lambda_A+2\lambda_B})w}\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)w}{(\lambda_A+2\lambda_B)^2}}\Big)dw\Big)\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\\
&\quad\times e^{-\frac{Hq(\lambda_A+\lambda_B)}{\lambda_A+2\lambda_B}}\cdot e^{\frac{\lambda_B^2Hz}{(\lambda_A+\lambda_B)(\lambda_A+2\lambda_B)}}\cdot\mathcal{I}_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\Big)\Big]dz\mathbf{1}_{(p,\infty)}(s).
\end{aligned}\tag{4.8}
$$

5 The Loss Probability

A further special case is to get the probability that player A loses to player B. This can be directly obtained from

$$\varphi_\mu(u,v,\theta)=E\Big[e^{-u\alpha_\mu-v\beta_\mu-\theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big]\tag{5.1}$$

by setting $u=v=\theta=0$:

$$\varphi_\mu(0,0,0):=E[\mathbf{1}_{\{\mu<\nu\}}]=P\{\mu<\nu\}=P\{t_\mu<t_\nu\}.\tag{5.2}$$

With $u = v = \theta = 0$ in (1.8), we have

***(i)* Case $\boldsymbol{\delta \neq \lambda_A}$,**

$$\begin{aligned}\varphi^1_\mu(0,0,0) &= \frac{-\lambda_A}{\lambda_A+\lambda_B}\cdot e^{-ph} + \frac{\lambda_A}{\lambda_A+\lambda_B}\cdot e^{-ph}\cdot e^{-q(\frac{H\delta}{\delta+\lambda_B})}\\ &+ \frac{-\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(\frac{H\delta}{\delta+\lambda_B})}\\ &+ \frac{\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}})\\ &+ \frac{\lambda_A H}{\lambda_A+\lambda_B}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\int_{z=0}^{q} e^{-(\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz\\ &+ \frac{-\lambda_A\lambda_B^2 H}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^2}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{H\delta}{\delta+\lambda_B})}\\ &\quad\times\int_{z=0}^{q} e^{(\frac{H\delta}{\delta+\lambda_B}-\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz. \end{aligned}\tag{5.3}$$

***(ii)* Case $\boldsymbol{\delta = \lambda_A}$,**

$$\begin{aligned}\varphi^2_\mu(0,0,0) &= \frac{-\lambda_A}{\lambda_A+\lambda_B}\cdot e^{-ph} + \frac{\lambda_A}{\lambda_A+\lambda_B}\cdot e^{-ph}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\\ &+ \frac{-\lambda_A^2}{(\lambda_A+\lambda_B)^2}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\\ &+ \frac{\lambda_A^2}{(\lambda_A+\lambda_B)^2}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}})\\ &+ \frac{\lambda_A H}{\lambda_A+\lambda_B}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\int_{z=0}^{q} e^{-(\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz\\ &+ \frac{-\lambda_A\lambda_B^2 H}{(\lambda_A+\lambda_B)^2}\sqrt{\frac{q}{\lambda_A\lambda_B hHp}}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_1(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}). \end{aligned}\tag{5.4}$$

6 Numerical Results

Since the above formulas may look a little bulky, some numerical results can well illustrate them and add to their credibility. They also show how changing input parameters alters the trend of the game. For the full completion of the demonstration we bring here a detailed MATLAB routine, which can be utilized for anyone wanting to run their own input parameters such as λ_A, λ_B, h, H, p, q and δ.

```
%The probability that player A loses to player B when delta is not equal to
%lambda A.
%A=lambda A, B=lambda B, d=delta

syms z
A=18, B=20, h=14, H=16, p=20, q=24, d=500;
```

```
f11=-A/(A+B)*exp(-p*h)+A/(A+B)*exp(-p*h)*exp(-q*H*d/(d+B))
-A*d/((A+B)*(d+B))*exp(-p*B*h/(A+B))*exp(-q*H*d/(d+B))
f12=A*d/((A+B)*(d+B))*exp(-p*B*h/(A+B))*exp(-q*A*H/(A+B))
*double(besseli(0,2*sqrt(A*B*h*H*p*q/(A+B)^2)))
f13=A*H/(A+B)*exp(-p*B*h/(A+B))*double(int(exp(-A*H*z/(A+B))
*besseli(0,2*sqrt(A*B*h*H*p*z/(A+B)^2)),0,q))
f14=-A*B^2*H/((A+B)*(d+B)^2)*exp(-p*B*h/(A+B)) *exp(-q*H*d/(d+B))
*double(int(exp((H*d/(d+B)-A*H/(A+B))*z)
*besseli(0,2*sqrt(A*B*h*H*p*z/(A+B)^2)),0,q))

Probability_A_Loses_B_1=f11+f12+f13+f14

%The probability that player A loses to player B when delta is equal to
%lambda A.
%A=lambda A, B=lambda B

syms z
A=10, B=5, h=24, H=12, p=30, q=25;

f21=-A/(A+B)*exp(-p*h)+A/(A+B)*exp(-p*h)*exp(-q*A*H/(A+B))
-A^2/(A+B)^2*exp(-p*B*h/(A+B))*exp(-q*A*H/(A+B))
f22=A^2/(A+B)^2*exp(-p*B*h/(A+B))*exp(-q*A*H/(A+B))
*double(besseli(0,2*sqrt(A*B*h*H*p*q/(A+B)^2)))
f23=A*H/(A+B)*exp(-p*B*h/(A+B))*double(int(exp(-A*H*z/(A+B))
*besseli(0,2*sqrt(A*B*h*H*p*z/(A+B)^2)),0,q))
f24=-A*B^2*H/(A+B)^2*sqrt(q/(A*B*h*H*p))*exp(-p*B*h/(A+B))
*exp(-q*A*H/(A+B))*double(besseli(1,2*sqrt(A*B*h*H*p*q/(A+B)^2)))

Probability_A_Loses_B_2=f21+f22+f23+f24
```

The program utilizes (5.3) and (5.4) and the calculations are put in the tables below.

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	20	18	17	16	14
H	16	16	16	16	16
p	20	20	20	20	20
q	24	24	24	24	24
δ	100	100	100	100	100
Probability of A losing	0.0733	0.3448	0.5591	0.7622	0.9711

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	14	14	14	14	14
H	16	14	13	12	10
p	20	20	20	20	20
q	24	24	24	24	24
δ	100	100	100	100	100
Probability of A losing	0.9711	0.7474	0.5070	0.2556	0.0181

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	14	14	14	14	14
H	16	16	16	16	16
p	28	26	24	22	20
q	24	24	24	24	24
δ	100	100	100	100	100
Probability of A losing	0.1064	0.3060	0.6027	0.8555	0.9711

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	14	14	14	14	14
H	16	16	16	16	16
p	20	20	20	20	20
q	24	24	24	24	24
δ	1	2	4	10	18
Probability of A losing	0.8904	0.9432	0.9605	0.9677	0.9695

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	14	14	14	14	14
H	16	16	16	16	16
p	20	20	20	20	20
q	24	24	24	24	24
δ	50	100	500	1,000	10,000
Probability of A losing	0.9708	0.9711	0.9714	0.9715	0.9715

λ_A	20	20	20	20	20
λ_B	18	18	18	18	18
h	10	12	13	14	16
H	14	14	14	14	14
p	24	24	24	24	24
q	20	20	20	20	20
δ	50	50	50	50	50
Probability of A losing	0.9811	0.7389	0.4864	0.2475	0.0278

λ_A	20	20	20	20	20
λ_B	18	18	18	18	18
h	16	16	16	16	16
H	14	16	17	18	20
p	24	24	24	24	24
q	20	20	20	20	20
δ	50	50	50	50	50
Probability of A losing	0.0278	0.2332	0.4350	0.6498	0.9247

λ_A	20	20	20	20	20
λ_B	18	18	18	18	18
h	16	16	16	16	16
H	14	14	14	14	14
p	24	24	24	24	24
q	20	20	20	20	20
δ	1	2	4	10	20
Probability of A losing	0.0124	0.0175	0.0218	0.0254	0.0269

where

$$
\begin{aligned}
\lambda_A^{-1} &= \text{The frequency of strikes to player A by player B,}\\
\lambda_B^{-1} &= \text{The frequency of strikes to player B by player A,}\\
h^{-1} &= \text{The average of magnitude of strikes to player A by player B,}\\
H^{-1} &= \text{The average of magnitude of strikes to player B by player A,}\\
p &= \text{The threshold of player A,}\\
q &= \text{The threshold of player B,}\\
\delta^{-1} &= \text{The observations frequency.}
\end{aligned}
$$

Concluding Remarks

In this paper, we continued our studies on fully antagonistic stochastic games of two players (A and B) (initiated in [3]), modeled by two independent marked Poisson processes. We investigated the paths in which player A loses the game. In this paper, we render calculation for a variety of special cases. The latter are presented either as fully explicit Laplace–Stieltjes joint transforms of the exit time and casualties to both players upon the exit or explicit probabilities and probability density functions, mostly in terms of modified Bessel functions. The results are illustrated by many numerical examples, and a MATLAB routine for calculation is attached.

References

[1] Bateman, H. and Erdélyi. *Tables of Integral Transforms*, Volume I, McGraw-Hill, 1954.

[2] Bateman, H. and Erdélyi. *Higher Transcedental Functions*, Volume 2, McGraw-Hill, 1953.

[3] Dshalalow, J.H. and Treerattrakoon, A. Antagonistic games with an initial phase. *Nonlinear Dynamics and Systems Theory* **9** (3) (2009) 277-286.